2017

Levi subalgebras of gl(5, R)

Manoj Lamichhane
University of Toledo

A Dissertation

entitled

Levi Subalgebras of $\mathfrak{gl}(5, \mathbb{R})$

by

Manoj Lamichhane

Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics

Dr. Gerard Thompson, Committee Chair

Dr. Akaki Tikaradze, Committee Member

Dr. Alessandro Arsie, Committee Member

Dr. Frans Cantrijn, Committee Member

Dr. Amanda Bryant-Friedrich, Dean
College of Graduate Studies

The University of Toledo
August 2016

An Abstract of

Levi Subalgebras of $\mathfrak{gl}(5, \mathbb{R})$

by

Manoj Lamichhane

Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics

The University of Toledo
August 2016

This dissertation concerns Levi subalgebras of $\mathfrak{gl}(5, \mathbb{R})$. All the simple subalgebras of $\mathfrak{gl}(5, \mathbb{R})$ are found together with their representations comprising seventeen in all. The semi-simple subalgebras of $\mathfrak{gl}(5, \mathbb{R})$ are then found together with their representations comprising six cases in total. Each such semi-simple subalgebra acts by commutator on $\mathfrak{gl}(5, \mathbb{R})$ and the resulting representations are fully decomposed. The results are used to determine all possible solvable extensions of a given semi-simple subalgebra and hence all Levi subalgebras of $\mathfrak{gl}(5, \mathbb{R})$ are determined up to isomorphism allowing also for the use of companion subalgebras.

To my parents Dilli Ram Upadhyaya and Chandrakala Upadhyaya

Acknowledgments

I would never have been able to complete my dissertation without the timely help and support of my advisor, Prof. Gerard Thompson. I would therefore like to express my sincere appreciation to Prof. Thompson for his invaluable guidance, availability and caring. His enthusiasm, encouragement, knowledge and attitude have always been in my favor during my graduate study time at University of Toledo.

I would like to thank my committee members Dr. Alessandro Arsie and Dr. Akaki Tikaradze for providing their honest feedback, being very friendly and supporting. I would also like to thank Dr. Frans Cantrijn from Ghent University for his willingness to be a outside reader of my committee.

Finally, I would like to thank my parents, my wife Sabitra, my children Aryav and Arvi, entire friends and family. Sabitra has always been there supporting and inspiring me through the both good times and bad times.

Contents

Chapter 1

Preliminary Material

1.1 Introduction

In this dissertation we consider the problem of classifying the Levi Lie subalgebras of $\mathfrak{gl}(5, \mathbb{R})$. By a *Levi algebra* we understand a Lie algebra that is a semi-direct product of a semi-simple algebra σ and a solvable algebra ρ. Details about the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ and $\mathfrak{gl}(3, \mathbb{R})$ may be found in [TW]. One could of course decide to classify the Lie subalgebras of $\mathfrak{sl}(5, \mathbb{R})$ instead of $\mathfrak{gl}(5, \mathbb{R})$, which has the advantage of avoiding many trivial and obvious subalgebras; however, the drawback is that in many other cases one has to impose a somewhat arbitrary condition on the parameters in order to obtain a subalgebra of $\mathfrak{sl}(5, \mathbb{R})$ rather than $\mathfrak{gl}(5, \mathbb{R})$ so we have opted to stick with the latter.

Every abstract semi-simple subalgebra is a direct sum of simple subalgebras. By "abstract" here we mean that the given subalgebra does not necessarily appear as a subalgebra of $\mathfrak{gl}(k, \mathbb{R})$ for some k, although Ado's Theorem informs us that such a k must exist. For small values of k it is feasible to find all the irreducible representations of such a simple subalgebra in $\mathfrak{gl}(k, \mathbb{R})$ and hence, up to isomorphism, all the representations of the semi-simple subalgebra, which are direct sums of irreducible representations.

Suppose that we start with a certain semi-simple subalgebra $\sigma \subset \mathfrak{gl}(n, \mathbb{R})$. Then σ acts on $\mathfrak{gl}(n, \mathbb{R})$ *as a Lie algebra* by $s(g) = [s, g]$; that is, commutator defines a Lie algebra homomorphism from σ into $\mathrm{End}(\mathfrak{gl}(n, \mathbb{R}))$, as readily follows from the Jacobi identity. Since σ is semi-simple, this representation must be completely reducible. To construct Levi subalgebras of $\mathfrak{gl}(n, \mathbb{R})$ we have to find the irreducible submodules or invariant subspaces of this representation. A particular submodule may or may not define a solvable subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. In order to find all possible Levi subalgebras corresponding to a particular a σ we have to see if it is possible to add any of the submodules together so as to define a larger solvable subalgebra.

As regards the invariant subspaces: $\mathfrak{sl}(2, \mathbb{R})$ (in one of its representations) is acting on $\mathfrak{gl}(5, \mathbb{R})$ by commutator. We are looking, ultimately, for Levi subalgebras of $\mathfrak{gl}(5, \mathbb{R})$ for which $\mathfrak{sl}(2, \mathbb{R})$ (in one of its representations) is the simple part. In a Levi decomposition, the semisimple part always acts by commutator on the radical, in other words, we get a representation of the semi-simple part. Since $\mathfrak{sl}(2, \mathbb{R})$ is simple, it must be possible to decompose the action: it must be completely reducible into a direct sum of irreducible components. Also, we know (up to isomorphism) all the possible representations of $\mathfrak{sl}(2, \mathbb{R})$. They are given in the 1988 paper by Turkowski [Tur1].

1.2 Constructing algebras with a non-trivial Levi decomposition

Let us consider the problem of constructing a Lie algebra that has a Levi decomposition $\sigma \rtimes \rho$ in general. We have the following structure equations:

$$[e_a, e_b] = C^c_{ab} e_c,\ [e_a, e_i] = C^k_{ai} e_k,\ [e_i, e_j] = C^k_{ij} e_k \tag{1.1}$$

where $1 \leq a, b, c, d \leq r$ and $r+1 \leq i, j, k, l \leq n$ and $\{e_a\}$ is a basis for the semi-simple subalgebra σ and $\{e_i\}$ is a basis for the radical ρ. Calculation shows that the Jacobi identity is equivalent to the following conditions:

$$C^e_{[ab}C^d_{c]e} = 0, \; C^c_{ab}C^j_{ci} = C^k_{bi}C^j_{ak} - C^k_{ai}C^j_{bk},$$

$$C^k_{al}C^l_{ij} = C^l_{ai}C^k_{lj} - C^l_{aj}C^k_{li}, \; C^l_{[ij}C^m_{k]l} = 0. \tag{1.2}$$

We interpret (1.2) as follows: we start with a semi-simple algebra so that the first set of conditions above is satisfied. Then the second set say that the matrices C^k_{al} make ρ (merely as a vector space) into a σ-module. The third set say that the C^k_{al} are derviations of the Lie algebra ρ and the fourth of course that ρ is a Lie algebra. Therefore to find all possible Lie algebras of dimension n that have a Levi decomposition $\sigma \rtimes \rho$ we can proceed as follows: choose a semi-simple algebra σ of dimension r. Then pick any solvable algebra ρ of dimension $n - r$ and consider a representation (should it exist) of σ in ρ considered simply as the vector space $\mathbb{R}^{n-r}$, all of which are known and are completely reducible [Hum] since σ is semi-simple. Finally, it only remains to check that the matrices representing σ act as derivations of the Lie algebra ρ. In the affirmative case we have our sought after Lie algebra; in the negative case there is no such algebra and we have to choose a different representation of σ in ρ. If all such representations lead to a null result then there can be no Levi decomposition involving σ and ρ.

1.3 Procedure

- Find all simple subalgebras of $\mathfrak{gl}(5, \mathbb{R})$.
- Find all possible representations (up to change of basis) of simple subalgebras of $\mathfrak{gl}(5, \mathbb{R})$.

- Find all semi-simple (not simple) subalgebras of $\mathfrak{gl}(5,\mathbb{R})$. It can only happen in six ways.
- Find Levi algebras by adding invariant subspaces coming from the action by commutator of the given semi-simple subalgebra acting on $\mathfrak{gl}(5,\mathbb{R})$ by commutator.

1.4 Classification of the low-dimensional Lie algebras

Having obtained all the semi-simple algebras of $\mathfrak{gl}(5,\mathbb{R})$ the task now is to find in each case the possible solvable radicals that can be added so as to produce a Levi subalgebra. In order to describe these radicals it will be helpful to describe the state of the art in the classification of the low-dimensional Lie algebras, a program that can be traced back to Lie himself. Such classifications have been attempted many times but are bedeviled by the fact that Lie algebras in general belong to continuous families and so the situation is much more complicated than for the semi-simple algebras. We shall follow the classifications obtained by G M Mubarakzyanov [Mub1, Mub2], who produced a list of the isomorphism classes of the real indecomposable Lie algebras up to dimension five that is generally accepted as being correct. That list is readily accessible in [PSWZ].

The situation in dimension six is altogether more complicated. As regards solvable indecomposable Lie algebras there are three classes, the nilpotent Lie algebras, and the algebras that have nilradicals of dimensions five and four, respectively. Morozov classified the six-dimensional nilpotent Lie algebras, a first version of which was produced by Umlauf [Um] as early as 1890. In fact Morozov's list was improved in [GTT] and it was shown that the six-dimensional nilpotents do not to belong to continuous families. Mubarakzyanov [Mub3] himself classified the solvable indecomposable Lie algebras of dimension six that have a five-dimensional nilradical and [ST] attempts to correct the obvious flaws in his work. Turkowski [Tur2] classified the solvable indecomposable Lie algebras of dimension six that have a four-dimensional nilradical. Finally in another article Turkowski [Tur1] classified the solvable indecomposable Levi algebras of dimensions between five and eight that have a non-trivial Levi decomposition. Following the original notation of the cited authors [Mub1, Mub2, Mub3, Tur1, Tur2], the algebras of dimension up to and including five will be denoted by $A_{k,m}$ where k denotes the dimension of the algebra and m the number in the author's list. A similar notation applies to the six-dimensional Mubarakzyanov algebras [Mub3] and $N_{k,m}$ for the solvable six-dimensional algebras classified by Turkowski [Tur2]. The Levi algebras classified by Turkowski [Tur1] will be denoted by $L_{k,m}$: the algebras $A_{5.40}$ and $L_{5.1}$ coincide. We shall not need to refer to the six-dimensional nilpotent algebras. Finally, the non-abelian two-dimensional algebra is denoted by $A_{2.1}$. We must mention also the recent encyclopaedic book by Snobl and Winternitz [SW] on the subject of the classification of the low-dimensional Lie algebras for many more references. For lack of space in Section 7 we are not able to give the specific isomorphism that identifies the radical; moreover, when we refer to an algebra such as $A_{4,5a=b=c=1}$ the letters pertain to the classifications obtained in the references given in this paragraph and *not* to the same letters that serve as coordinates for a particular subalgebra that occurs in our list. In the construction of the Levi algebras, various nilpotent subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ occur and they are listed in the Appendix.

Chapter 2

Theory

This Chapter will be concerned with most of the theory that will be needed throughout this dissertation.

2.0.1 Subalgebras of $\mathfrak{gl}(2,\mathbb{R})$

The first topic that we need to address concerns subalgebras of $\mathfrak{gl}(2,\mathbb{R})$. For more details the reader may consult [TW] where the *subgroups* of $GL(2,\mathbb{R})$ were found.

Lemma 2.0.1. *Let $\mathfrak{g}$ be an abelian two-dimensional subalgebra of $\mathfrak{gl}(2,\mathbb{R})$. Then there exists a basis for $\mathfrak{g}$ given by $\{A,B\}$ where $A=\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], B=\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ or $A=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], B=\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ or $A=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], B=\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$.*

Lemma 2.0.2. *Let $\mathfrak{g}$ be a non-abelian two-dimensional subalgebra of $\mathfrak{gl}(2,\mathbb{R})$. Then there exists a basis for $\mathfrak{g}$ given by $\{A,B\}$ where $A=\left[\begin{smallmatrix}\lambda+1&0\\0&\lambda\end{smallmatrix}\right], B=\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ and $\lambda\in\mathbb{R}$.*

In fact all the subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ can be listed as [TW]:

$$\begin{bmatrix}\epsilon x & x\\ 0 & \epsilon x\end{bmatrix}\begin{bmatrix}x & 0\\ 0 & \lambda x\end{bmatrix}\begin{bmatrix}\lambda x & x\\ -x & \lambda x\end{bmatrix}\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}\begin{bmatrix}x & y\\ 0 & x\end{bmatrix}\begin{bmatrix}x & y\\ -y & x\end{bmatrix}\begin{bmatrix}(\lambda+1)x & y\\ 0 & \lambda x\end{bmatrix}\begin{bmatrix}x & z\\ 0 & y\end{bmatrix}$$

where x, y, z are coordinates on a given subalgebra, to which have to be added $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{gl}(2,\mathbb{R})$ itself.

2.1 Subalgebras of $\mathfrak{gl}(3,\mathbb{R})$

2.1.1 One-dimensional algebras

To obtain the one-dimensional algebras simply take the real Jordan Normal Forms for 3×3 matrices and we obtain the following possibilities:

Lemma 2.1.1. *Over $\mathbb{R}$ every 3×3 matrix is equivalent under change of basis to one of the following:*

$$a)\begin{bmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{bmatrix}\quad b)\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{bmatrix}\quad c)\begin{bmatrix}\alpha & 1 & 0\\ -1 & \alpha & 0\\ 0 & 0 & \beta\end{bmatrix} d)\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}.$$

2.1.2 Two-dimensional subalgebras of $\mathfrak{gl}(3,\mathbb{R})$

2.1.2.1 Abelian subalgebras

Lemma 2.1.2. *Let $\mathfrak{g}$ be an abelian two-dimensional subalgebra of $\mathfrak{gl}(3,\mathbb{R})$. Then we may choose a basis for $\mathfrak{g}$ in one of the following six forms:*

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & a\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & b\end{bmatrix};\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & \epsilon\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & c\end{bmatrix};\begin{bmatrix}\epsilon & 1 & 0\\ 0 & \epsilon & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix};$$
$$\begin{bmatrix}\epsilon & 1 & 0\\ 0 & \epsilon & 0\\ 0 & 0 & \epsilon\end{bmatrix}\begin{bmatrix}\epsilon & 0 & c\\ 0 & \epsilon & 0\\ 0 & e & \epsilon\end{bmatrix};\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & b\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & c\end{bmatrix};\begin{bmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & c\end{bmatrix}.$$

Proof. We know that up to isomorphism there are just two "abstract" Lie algebras of dimension two, the abelian algebra $\mathbb{R}^2$ and the one for which $[e_1,e_2]=e_1$. To find the abelian subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ we take a matrix A of the form found in Section 3. Then we find the most general matrix B that commutes with it and then normalize the entries of B.

1. To begin with let A be a matrix of type a) in Section 3. We may assume that both matrices are of type a): if not one of the matrices is of type b),c) or d), each of which will be considered below in turn. Since two commuting diagonalizable matrices are simultaneously diagonalizable we can reduce to

$$a)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix}$$

2. Now let A be a matrix of type b) in Section 3. We have to distinguish two cases depending on whether $\lambda \neq \mu$ or $\lambda = \mu$. Suppose the former case holds and that the matrix is called A. Then the most general matrix that commutes with A is of the form $B = \left[\begin{smallmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & d \end{smallmatrix}\right]$. Then considering $\overline{B} := bA - B$ reduces b to zero. If $a \neq 0$ we can reduce a to unity and λ to zero and μ to zero or unity. If, however, $a = 0$ we can assume that $d = 1$ and that $\lambda = 0, 1$. To summarize, in this subcase we have:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} (\mu = 0, 1) \qquad \begin{bmatrix} \epsilon & 1 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\lambda = 0, 1)$$

Now suppose that A is a matrix of type b) in Section 3 and that $\lambda = \mu$. Then the most general matrix that commutes with A is of the form $B = \left[\begin{smallmatrix} a & b & c \\ 0 & a & 0 \\ 0 & e & d \end{smallmatrix}\right]$. Now if $a \neq d$ then the matrix $\begin{bmatrix} 1 & 0 & -\frac{c}{d-a} \\ 0 & 1 & 0 \\ 0 & \frac{e}{d-a} & 1 \end{bmatrix}$ preserves A and reduces c and e in B to zero. Thus we are reduced to the case of the last paragraph.

The only remaining case is where both matrices are of type b) in Section 3 with $\lambda = \mu$. In this case we can easily reduce to

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} (\lambda = 0, 1) \begin{bmatrix} a & 0 & c \\ 0 & a & 0 \\ 0 & e & a \end{bmatrix} (a = 0, 1).$$

If we permute the basis vectors e_2 and e_3 then we obtain the matrices in upper triangular form.

$$\begin{bmatrix} \epsilon & 0 & 1 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \begin{bmatrix} \epsilon & c & 0 \\ 0 & \epsilon & e \\ 0 & 0 & \epsilon \end{bmatrix}.$$

3. For a matrix A of type c) in Section 3 the only possibility for a matrix B that commutes with A is $B = \left[\begin{smallmatrix} \gamma & \delta & 0 \\ -\delta & \gamma & 0 \\ 0 & 0 & \rho \end{smallmatrix}\right]$. Now we can assume that $\delta = 0$ and then either $\alpha = 0$ and $\gamma = 1$ or $\gamma = 0, \rho = 1$ and $\beta = 0$.

4. We conclude with a matrix of type d) in Section 3 that we call A. It has only one Jordan block and so in particular is "non-derogatory". The only matrices that commute with it are powers of A or equivalently are of the form $B = \left[\begin{smallmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{smallmatrix}\right]$. If $\lambda \neq 0$ by taking linear combinations and scaling we may assume that $\lambda = 1$ and $a = 0$. Note that if we divide A by λ we may conjugate by a diagonal matrix so as to restore the $(1, 2)$ and $(2, 3)$ entries to unity. We can now scale B by $\frac{1}{b}$ so as to reduce to $b = 1$ if $b \neq 0$. If, however, $b = 0$ then $c \neq 0$ and we can scale c to unity.

   If $\lambda = 0$ but $a \neq 0$ we can assume that $b \neq 0$ and then we can reduce the second generator to type d) in Section 3 for which $\lambda \neq 0$, a case which has just been considered. Finally if $\lambda = a = 0$ or then we can reduce to $b = 0, c = 1$. We can summarize these cases as:

$$a) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad b) \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} (a = 0, 1) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

2.1.2.2 Non-abelian two-dimensional subalgebras

The following argument appears in [TW]. We write the algebra as $[A, B] = A$. Then we take in turn A to be a matrix of type a)b)c)d) that appears in Section 3. It turns out that in the case of a) and c) there is no solution because of the fact the commutator of an arbitrary matrix and a diagonal matrix has zeroes on the diagonal.

For a matrix of type b) we find that necessarily $\lambda = \mu = 0$. Then we can subtract a multiple of A from B so as to obtain for A and B

$$b) \, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \, B = \begin{bmatrix} a & 0 & c \\ 0 & a+1 & 0 \\ 0 & d & e \end{bmatrix}.$$

If $a \neq e$ we can assume that $c = 0$ and if $a - e + 1 \neq 0$ that $d = 0$. Furthermore in these last two cases we may assume by change of basis that $d = 0, 1$ and $c = 0, 1$ giving the following three cases:

$$\begin{bmatrix} ax & y & 0 \\ 0 & (a+1)x & 0 \\ 0 & 0 & bx \end{bmatrix} \begin{bmatrix} ax & y & x \\ 0 & (a+1)x & 0 \\ 0 & 0 & ax \end{bmatrix} \begin{bmatrix} ax & y & 0 \\ 0 & (a+1)x & 0 \\ 0 & 0 & (a+1)x \end{bmatrix}.$$

For type d) It turns out that necessarily $\lambda = 0$. The solution is

$$d)\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} b & c & d \\ 0 & b+1 & c \\ 0 & 0 & b+2 \end{bmatrix}.$$

However, by change of basis that fixes A, it is possible to reduce c to zero.

2.1.3 Subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ of dimension three and higher

In [TW] the reader may find a list of subgroups of $GL(3,\mathbb{R})$ corresponding to every possible subalgebra of $\mathfrak{gl}(3,\mathbb{R})$; here we shall provide a much simplified list of the solvable subalgebras of $\mathfrak{gl}(3,\mathbb{R})$ the cases of dimension one and two having already been given in Lemmas 2.1.1 and 2.1.2.

2.1.3.1 Dimension 3

$A_{3.1}$ $[e_2, e_3] = e_1$:

$$\begin{bmatrix} \alpha x + \beta y & x & z \\ 0 & \alpha x + \beta y & y \\ 0 & 0 & \alpha x + \beta y \end{bmatrix}.$$

$A_{3.2}$ $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$:

$$\begin{bmatrix} (\alpha+1)z & z & x \\ 0 & (\alpha+1)z & y \\ 0 & 0 & \alpha z \end{bmatrix}.$$

$A_{3.3,3.4,3.5a(a\neq\pm1)}$ $[e_1, e_3] = e_1, [e_2, e_3] = ae_2$:

$$\begin{bmatrix} (\beta+1)z & 0 & x \\ 0 & (a+\beta)z & y \\ 0 & 0 & \beta z \end{bmatrix}.$$

$A_{3.6(a=0),3.7a(a\neq 0)}$ $[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2$:

$$\begin{bmatrix} (a+\beta)z & z & x \\ -z & (a+\beta)z & y \\ 0 & 0 & \beta z \end{bmatrix}.$$

$\mathbb{R}^3$

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \begin{bmatrix} z & 0 & x \\ 0 & z & y \\ 0 & 0 & z \end{bmatrix}.$$

$A_{2.1} \oplus \mathbb{R}$

$$\begin{bmatrix} \alpha x + \lambda z & y & 0 \\ 0 & (\alpha+1)x + \lambda z & 0 \\ 0 & 0 & \beta x + \mu z \end{bmatrix} \begin{bmatrix} \alpha x + \lambda z & y & 0 \\ 0 & (\alpha+1)x + \lambda z & 0 \\ 0 & 0 & \alpha x + \mu z \end{bmatrix}$$

$$\begin{bmatrix} \alpha x + \lambda z & y & x + \mu z \\ 0 & (\alpha+1)x + \lambda z & 0 \\ 0 & 0 & \alpha x + \lambda z \end{bmatrix} \begin{bmatrix} \alpha x + \lambda z & y & 0 \\ 0 & (\alpha+1)x + \lambda z & 0 \\ 0 & 0 & (\alpha+1)x + \mu z \end{bmatrix}$$

$$\begin{bmatrix} \alpha x + \lambda z & y & 0 \\ 0 & (\alpha+1)x + \lambda z & 0 \\ 0 & x + \mu z & (\alpha+1)x + \lambda z \end{bmatrix} \begin{bmatrix} x & y & 0 \\ 0 & x + z & y \\ 0 & 0 & x + 2z \end{bmatrix}.$$

2.1.3.2 Dimension 4

If an algebra is decomposable it must decompose, as an abstract algebra, as a sum of either $\mathbb{R}$ and a three-dimensional indecomposable, a sum of a two-dimensional indecomposable and two copies of $\mathbb{R}$, or a sum of two two-dimensional indecomposables or be abelian. However, the latter alternative cannot occur. There is no four-dimensional abelian subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ according to the Schur-Jacobson result [Jac].

$A_{4.8,4.9b(-1<b\leq 1)}[e_2,e_3]=e_1,[e_1,e_4]=(b+1)e_1,[e_2,e_4]=e_2,[e_3,e_4]=be_3$:

$$\begin{bmatrix} (\alpha+b+1)w & x & z \\ 0 & (\alpha+b)w & y \\ 0 & 0 & \alpha w \end{bmatrix}.$$

$A_{4.12}[e_1,e_3]=e_1,[e_2,e_3]=e_2,[e_1,e_4]=-e_2,[e_2,e_4]=e_1$

$$\begin{bmatrix} \alpha z+\beta w & w & x \\ -w & \alpha z+\beta w & y \\ 0 & 0 & (\alpha+1)z+\beta w \end{bmatrix}.$$

$A_{2.1}\oplus\mathbb{R}^2$, $[e_1,e_2]=e_2$:

$$\begin{bmatrix} x & w & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}\begin{bmatrix} \alpha x+z & y & w \\ 0 & (\alpha+1)x+z & 0 \\ 0 & 0 & \alpha x+z \end{bmatrix}.$$

$A_{2.1}\oplus A_{2.1},[e_1,e_2]=e_2,[e_3,e_4]=e_4$

$$\begin{bmatrix} \alpha x+\lambda z & y & w \\ 0 & (\alpha+1)x+\lambda z & 0 \\ 0 & 0 & \alpha x+\mu z \end{bmatrix}.$$

$\mathbb{R}\oplus A_{3.1}[e_2,e_3]=e_1$:

$$\begin{bmatrix} w+\alpha x+\beta y & x & z \\ 0 & w+\alpha x+\beta y & y \\ 0 & 0 & w+\alpha x+\beta y \end{bmatrix}.$$

$\mathbb{R}\oplus A_{3.2}$, $[e_1,e_3]=e_1,[e_2,e_3]=e_1+e_2$:

$$\begin{bmatrix} w+(\alpha+1)z & z & x \\ 0 & w+(\alpha+1)z & y \\ 0 & 0 & w+\alpha z \end{bmatrix}.$$

$\mathbb{R}\oplus A_{3.3}, \mathbb{R}\oplus A_{3.4}, \mathbb{R}\oplus A_{3.5}(a\neq \pm 1)$ $[e_1,e_3]=e_1, [e_2,e_3]=ae_2$:

$$\begin{bmatrix} w+(\beta+1)z & 0 & x \\ 0 & w+(a+\beta)z & y \\ 0 & 0 & w+\beta z \end{bmatrix}.$$

$\mathbb{R}\oplus A_{3.6a=0}, \mathbb{R}\oplus A_{3.7a(a\neq 0)}[e_1,e_3]=ae_1-e_2, [e_2,e_3]=e_1+ae_2$:

$$\begin{bmatrix} w+(a+\beta)z & z & x \\ -z & w+(a+\beta)z & y \\ 0 & 0 & w+\beta z \end{bmatrix}.$$

2.1.4 Dimension 5

$A_{5.36}[e_2,e_3]=e_1, [e_1,e_4]=e_1, [e_2,e_4]=e_2, [e_2,e_5]=-e_2, [e_3,e_5]=e_3$:

$$\begin{bmatrix} (\alpha+1)w+\beta q & x & z \\ 0 & \alpha w+(\beta+1)q & y \\ 0 & 0 & \alpha w+\beta q \end{bmatrix}.$$

$\mathbb{R}\oplus A_{4.8,4.9b(-1<b\leq 1)}[e_2,e_3]=e_1, [e_1,e_4]=(b+1)e_1, [e_2,e_4]=e_2, [e_3,e_4]=be_3$:

$$\begin{bmatrix} q+(b+1)w & x & z \\ 0 & q+bw & y \\ 0 & 0 & q \end{bmatrix}.$$

$\mathbb{R}\oplus A_{4.12}[e_1,e_3]=e_1, [e_2,e_3]=e_2, [e_1,e_4]=-e_2, [e_2,e_4]=e_1$

$$\begin{bmatrix} z & w & x \\ -w & z & y \\ 0 & 0 & q \end{bmatrix}.$$

$A_{2.1}\oplus A_{3.3}$ $[e_1,e_2]=e_2, [e_3,e_5]=e_3, [e_4,e_5]=e_4$

$$\begin{bmatrix} \alpha q+x & w & z \\ 0 & (\alpha+1)q+x & 0 \\ 0 & 0 & \alpha q+y \end{bmatrix}.$$

$\mathbb{R} \oplus A_{2.1} \oplus A_{2.1}$ $[e_2, e_3] = e_2, [e_4, e_5] = e_4$

$$\begin{bmatrix} x & q & w \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}.$$

2.1.5 Dimension 6

$\mathbb{R} \oplus A_{5.36}[e_2, e_3] = e_1, [e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_2, e_5] = -e_2, [e_3, e_5] = e_3$

$$\begin{bmatrix} w & x & y \\ 0 & z & p \\ 0 & 0 & q \end{bmatrix}.$$

2.2 Several Propositions

We quote the following result from [Hum].

Proposition 2.2.1. *Let V be a finite-dimensional vector space over a closed field of characteristic zero and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a non-zero Lie algebra acting irreducibly on V. Then $\mathfrak{g} \approx \sigma \oplus \rho$ where σ is semi-simple and ρ consists of multiples of the identity.*

We can also apply this result to real Lie algebras provided that in tensoring with $\mathbb{C}$, the original representation does not become reducible.

Proposition 2.2.2. *Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a non-zero Lie algebra of the form $\mathfrak{g} \approx \sigma \rtimes \rho$ where σ is semi-simple and ρ is solvable. If σ acts irreducibly on ρ then ρ must be abelian.*

Indeed, σ acts on the derived algebra $[\rho, \rho]$ and if σ acts irreducibly it can only be that $[\rho, \rho] = 0$ since $[\rho, \rho] \neq \rho$ because ρ is solvable. Thus, we can be sure that when we decompose the action of σ on $\mathfrak{gl}(n, \mathbb{R})$ into irreducible submodules, those submodules which correspond to subalgebras must be abelian.

Proposition 2.2.3. *Suppose that σ is a simple Lie algebra and that it has an irreducible representation in $\mathfrak{gl}(k, \mathbb{R})$. Suppose that we extend the representation by adding zero rows and zero columns so as to obtain a representation in $\mathfrak{gl}(n, \mathbb{R})$. Then:*

(i) The only matrices that commute with the $n \times n$ matrices that represent σ are of the form $\left[\begin{smallmatrix} \lambda I_k & 0 \\ 0 & B \end{smallmatrix}\right]$ where $\lambda \in \mathbb{R}$ and $B \in \mathfrak{gl}(n-k, \mathbb{R})$ so that such matrices comprise the kernel when this representation of σ acts by commutator on $\mathfrak{gl}(n, \mathbb{R})$.

(ii) Each part of the $(k+1)$th to n columns with entries from 1 to k in $\mathfrak{gl}(n, \mathbb{R})$ and each part of the $(k+1)$th to n rows with entries from 1 to k comprise minimal invariant subspaces.

Proposition 2.2.4. *Suppose that in the previous Proposition σ is $\mathfrak{sl}(k, \mathbb{R})$ in its definition representation. Then $ad(\sigma)$ is a representation in $\mathbb{R}^{n^2}$ consisting of the generalized inflation of $ad(\mathfrak{sl}(k, \mathbb{R}))$ and $2(n-k)$ copies of the definition representation and a kernel of dimension $(n-k)^2 + 1$.*

Proposition 2.2.5. *Let $S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ be a block diagonal matrix where A is $p \times p$ and B is $q \times q$ and both A and B correspond to an irreducible* independent *representations over $\mathbb{C}$, that is, we have a direct sum representation. Then the only matrices that commute with S are of the form $\begin{bmatrix} \lambda I_p & 0 \\ 0 & \mu I_q \end{bmatrix}$ where A is $p \times p$ and B is $q \times q$ unless $p = q$ in which case matrices of the following form commute with S: $\begin{bmatrix} \lambda I_p & \nu I_p \\ \rho I_p & \mu I_q \end{bmatrix}$.*

Proof. Consider a matrix of the form $T = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ where W is $p \times p$, X is $p \times q$, Y is $q \times p$ and Z is $q \times q$. Working out the commutator and setting it to zero gives

$$[A, W] = 0, \quad AX - XB = 0, \quad BY - YA = 0, \quad [B, Z] = 0.$$

Since A and B correspond to an irreducible representations over $\mathbb{C}$ we can only have $W = \lambda I_p$ and $Z = \lambda I_q$. The other two conditions assert that X and Y are intertwining operators between irreducible representations. Schur's Lemma implies that we can only have $X = 0$ and $Y = 0$ unless the representations are equivalent in which case $p = q$ $X = \rho I_p$ and $Y = \sigma I_p$ for some ρ and $= \sigma$. □

Proposition 2.2.6. *Let* $S = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ *be a two-fold inflation of* A *which is* $p \times p$ *and corresponds to an irreducible* independent *representation over* $\mathbb{C}$*. Then the only matrices that commute with* S *are of the form* $\begin{bmatrix} \lambda I_p & \rho I_p \\ \sigma I_p & \mu I_q \end{bmatrix}$.

Proof. The proof is easily obtained by modifying the previous Proposition. □

2.2.1 Example

It will be useful to study an example. We consider in Proposition 2.2.4 $k = 2$ and $n = 5$. We have one copy of $ad(\mathfrak{sl}(2, \mathbb{R}))$, 6 copies of 2×2 $\mathfrak{sl}(2, \mathbb{R})$ and a 10-dimension kernel.

2.2.2 Technical Lemma

Lemma 2.2.1. *A pair of matrices in* $\mathfrak{gl}(n + 1, \mathbb{R})$ *of the form*

$$\begin{bmatrix} 0 & a \\ d^t & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & u \\ x^t & 0 \end{bmatrix}$$

can never be such that their commutator is of the form

$$\begin{bmatrix} \lambda I_n & 0 \\ 0 & \mu \end{bmatrix}$$

except in the trivial case where either $d = x = 0$ *or* $a = u = 0$ *and* $\lambda = \mu = 0$ *or* $u = \rho a$ *and* $x = \rho d$*.*

Proof. The commutator of the two matrices is

$$\begin{bmatrix} ax^t - ud^t & 0 \\ 0 & d^t u - x^t a \end{bmatrix}.$$

We can only have $\mu = -n\lambda$. By multiplying a and u by the same factor we can reduce to the cases $\lambda = \mu = 0$ or $\lambda = 1$ and $\mu = -n$.

In the first case we obtain $x^t da - d^t du = 0$ and $x^t xa - d^t xu = 0$. Unless $d = 0$ and $x = 0$ either a is a multiple of u or u is a multiple of a. Similarly either d is a multiple of x

or x is a multiple of d. By symmetry we have to consider the subcases $u = \alpha a, x = \beta d$ and $u = \alpha a, d = \beta x$. In the first case we find that $ax^t - ud^t = (\beta - \alpha)ad^t = 0$ and hence either $\beta = \alpha$ or a or $d = x = 0$. In the second case $ax^t - ud^t = (1 - \alpha\beta)ad^t = 0$ and hence either $\beta\alpha = 1$ or a or $d = x = 0$.

In the second case we obtain $ax^t - ud^t = I$ and $x^t a - d^t u = n$ and hence $x^t ua - d^t uu = u$ and $x^t aa - d^t au = a$. Thus a and u are linearly dependent unless $x^t u = 1 + d^t uu = x^t - 1 = d^t = 0$. However, these latter conditions imply that $n = 2$. Similarly d and x are linearly dependent. Thus there exist ρ, σ such that $u = \rho a$ and $x = \sigma d$; but now we find that $(\sigma - \rho)ad^t = I$. In this either $\sigma = \rho$ or $a = 0$ and hence $u = 0$ or $d = 0$ and hence $x = 0$. $\square$

2.2.3 Companion subalgebras

The following Proposition is based on [GT].

Proposition 2.2.7. *Suppose that the Lie algebra $\mathfrak{g}$ has a representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$. Suppose that $L : \mathfrak{gl}(p, \mathbb{R}) \to \mathfrak{gl}(p, \mathbb{R})$ is a (linear) involution, that is, has period two. Then mapping a representing matrix M to $-LM^tL$ gives a second inequivalent representation of $\mathfrak{g}$.*

Proof. Given $M \in \mathfrak{gl}(p, \mathbb{R})$ a representing matrix for $\mathfrak{g}$, map it to $\phi(M) = -LM^tL$. For a second such matrix N we have $[\phi(M), \phi(N)] = [-LM^tL, -LN^tL] = [LM^tLLN^tL - LN^tLLM^tL] = [LM^tN^tL - LN^tM^tL] = L[N, M]^tL = \phi([M, N])$. $\square$

There are two important special cases of the Proposition.

Corollary 2.2.2. *Suppose that the Lie algebra $\mathfrak{g}$ has a representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$. Then transposing the representing matrices about the diagonal (and taking negatives) gives a second inequivalent (that is not necessarily equivalent) representation of $\mathfrak{g}$.*

Corollary 2.2.3. *Suppose that the Lie algebra $\mathfrak{g}$ has a representation as a subalgebra of $\mathfrak{gl}(p, \mathbb{R})$. Then transposing the representing matrices about the anti-diagonal (and taking negatives) gives a second inequivalent (that is not necessarily equivalent) representation of $\mathfrak{g}$.*

Proof. Take for L in the Proposition the matrix whose only non-zero entries are 1's down the anti-diagonal. Then the map ϕ consists of taking a negative and transposing about the anti-diagonal. $\square$

Given a subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n, \mathbb{R})$ we shall define an algebra obtained by taking the negative transpose about the main or anti-diagonal as a *companion* of $\mathfrak{h}$. One should note that these companion subalgebras are not usually conjugate. In the interests of efficiency, when giving the list of the Levi algebras in Section 7 we shall not give the companion subalgebras but we shall be content to list algebras that are block upper triangular to the extent possible.

2.2.4 The Lattice of invariant subspaces

Let V be a $\mathfrak{g}$-module where $\mathfrak{g}$ is a given Lie algebra. A subspace $W \subset V$ is said to be $\mathfrak{g}$-invariant if for $S \in \mathfrak{g}$ we have $\phi(S)W \subset V$ where $\phi : \mathfrak{g} \to End(V)$ is the Lie algebra homomorphism of the module. The space of all invariant subspaces forms a lattice: the meet of two invariant subspaces V_1, V_2 is the intersection $V_1 \cap V_2$ and the join $V_1 + V_2$ where the sum is not necessarily direct. We call an invariant subspace *minimal* if it has no non-trivial invariant subspaces. Clearly V can be decomposed into a direct sum of minimal invariant subspaces.

It is important to understand that when a $\mathfrak{g}$-module has been decomposed into minimal invariant subspaces it is not necessarily true that every subspace is a sum of such minimal invariant subspaces. See for example the subalgebra $\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{3.1}$ in Chapter 4.

2.2.5 $\mathfrak{so}(4)$

A very interesting example is provided by $\mathfrak{so}(4)$ in its definition representation. For all values of $n \geq 3$, *except* $n = 4$, $\mathfrak{so}(n)$ is a simple Lie algebra. However, $\mathfrak{so}(4)$ is semi-simple and the following pair of matrices gives a decomposition of it as $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$:

$$\begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{bmatrix} \begin{bmatrix} 0 & d & e & f \\ -d & 0 & -f & e \\ -e & f & 0 & -d \\ -f & -e & d & 0 \end{bmatrix}.$$

Taking the first matrix above gives a representation of $\mathfrak{so}(3)$ in $\mathfrak{gl}(4, \mathbb{R})$. Although this representation is irreducible it becomes reducible if we complexify: in fact

$$\begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{bmatrix} = P^{-1} \begin{bmatrix} \sqrt{(-1)}a & 0 & 0 & b - \sqrt{(-1)}c \\ 0 & \sqrt{(-1)}a & b + \sqrt{(-1)}c & 0 \\ 0 & -b + \sqrt{(-1)}c & \sqrt{(-1)}a & 0 \\ -b - \sqrt{(-1)}c & 0 & d & \sqrt{(-1)}a \end{bmatrix} P$$

where

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \sqrt{(-1)} & -\sqrt{(-1)} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \sqrt{(-1)} & -\sqrt{(-1)} \end{bmatrix}.$$

So in the new basis $\langle e_1, e_4 \rangle$ and $\langle e_2, e_3 \rangle$ are each invariant subspaces.

Chapter 3

Semi-simple algebras of $\mathfrak{gl}(5,\mathbb{R})$ and their representations

3.1 Simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$

The real *simple* subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ consist, of $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{sl}(4,\mathbb{R})$, $\mathfrak{sl}(5,\mathbb{R})$, $\mathfrak{so}(3)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(4,1)$, $\mathfrak{so}(3,2)$, $\mathfrak{so}(5)$, $\mathfrak{sp}(4)$. In fact $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ have several inequivalent representations in $\mathfrak{gl}(5,\mathbb{R})$. In the cases of $\mathfrak{so}(4,1), \mathfrak{so}(5)$ and $\mathfrak{sl}(5,\mathbb{R})$ each of them has a representation in $\mathfrak{gl}(5,\mathbb{R})$ coming from its very definition and no smaller dimensional representation. As regards $\mathfrak{sl}(4,\mathbb{R})$ and $\mathfrak{so}(3,1)$, the only way to represent them in $\mathfrak{gl}(5,\mathbb{R})$ is to add an extra zero row and zero column to the definition representation and likewise for the definition representation of $\mathfrak{sp}(4)$. In the case of $\mathfrak{sl}(3,\mathbb{R})$ two extra zero rows and zero columns have to be added. It turns out that there is isomorphism between $\mathfrak{so}(3,2)$ and $\mathfrak{sp}(4)$ [Hel]; the definition gives us an irreducible representation of $\mathfrak{so}(3,2)$ and for $\mathfrak{sp}(4)$ we have to add an extra zero row and zero column. Altogether we find that there are, up to isomorphism, 17 simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ taking into account inequivalent representations.

3.2 Simple algebras of $\mathfrak{gl}(5,\mathbb{R})$ and their representations

3.2.1 $\mathfrak{sl}(2,\mathbb{R})$

$$I \quad \begin{bmatrix} 4a & 4b & 0 & 0 & 0 \\ c & 2a & 3b & 0 & 0 \\ 0 & 2c & 0 & 2b & 0 \\ 0 & 0 & 3c & -2a & b \\ 0 & 0 & 0 & 4c & -4a \end{bmatrix}$$

$$II \quad \begin{bmatrix} 3a & 3b & 0 & 0 & 0 \\ c & a & 2b & 0 & 0 \\ 0 & 2c & -a & b & 0 \\ 0 & 0 & 3c & -3a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$III \quad \begin{bmatrix} 2c & 2a & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 \\ 0 & 2b & -2c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$IV \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$VI \quad \begin{bmatrix} 2a & 2b & 0 & 0 & 0 \\ c & 0 & b & 0 & 0 \\ 0 & 2c & -2a & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & -a \end{bmatrix}$$

3.2.2 $\mathfrak{so}(3)$

$$VII \quad \begin{bmatrix} 0 & c & -b & 0 & 0 \\ -c & 0 & a & 0 & 0 \\ b & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$VIII \quad \begin{bmatrix} 0 & -a & -b & -c & 0 \\ a & 0 & -c & b & 0 \\ b & c & 0 & -a & 0 \\ c & -b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$IX \quad \begin{bmatrix} 0 & -2c & -2b & -2a & 0 \\ 2c & 0 & 2a & -2b & 0 \\ \frac{1}{2}b & -\frac{1}{2}a & 0 & -c & 3a \\ \frac{1}{2}a & \frac{1}{2}b & c & 0 & 3b \\ 0 & 0 & -a & -b & 0 \end{bmatrix},$$

3.2.3 $\mathfrak{sl}(3, \mathbb{R})$

$$X \begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & -(a+e) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.3.1 $\mathfrak{sl}(4, \mathbb{R})$

$$XI \begin{bmatrix} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & m & 0 \\ n & p & q & -(a+f+k) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.3.2 $\mathfrak{so}(3, 1)$

$$XII \quad \begin{bmatrix} 0 & a & b & c & 0 \\ -a & 0 & d & e & 0 \\ -b & -d & 0 & f & 0 \\ c & e & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.3.3 $\mathfrak{sp}(4)$

$$XIII \begin{bmatrix} a & b & e & f & 0 \\ c & d & f & g & 0 \\ h & i & -a & -c & 0 \\ i & j & -b & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.2.3.4 $\mathfrak{sl}(5, \mathbb{R})$

$$XIV \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & m & n & p & q \\ r & s & t & u & v \\ w & x & y & z & -(a+g+n+u) \end{bmatrix}.$$

3.2.3.5 $\mathfrak{so}(5)$

$$XV \quad \begin{bmatrix} 0 & a & b & c & g \\ -a & 0 & d & e & h \\ -b & -d & 0 & f & i \\ -c & -e & -f & 0 & j \\ -g & -h & -i & -j & 0 \end{bmatrix}.$$

3.2.3.6 $\mathfrak{so}(4,1)$

$$XVI \quad \begin{bmatrix} 0 & a & b & c & g \\ -a & 0 & d & e & h \\ -b & -d & 0 & f & i \\ -c & -e & -f & 0 & j \\ g & h & i & j & 0 \end{bmatrix}.$$

3.2.3.7 $\mathfrak{so}(3,2)$

$$XVII \quad \begin{bmatrix} 0 & a & b & c & g \\ -a & 0 & d & e & h \\ -b & -d & 0 & f & i \\ c & e & f & 0 & j \\ g & h & i & j & 0 \end{bmatrix}.$$

3.3 Semi-simple not simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$

Now we look for semi-simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ that are not simple: each such subalgebra must be a direct sum of simple subalgebras. According to Proposition 2.2.1 the only way to extend $\mathfrak{sl}(5,\mathbb{R})$, $\mathfrak{so}(3,2)$, $\mathfrak{so}(4,1)$ and $\mathfrak{so}(5)$ is to add multiples of the identity since the definition representation is irreducible; hence none of these subalgebras can occur as a summand in some semi-simple subalgebra of $\mathfrak{gl}(5,\mathbb{R})$. Also, it follows from Proposition 2.2.3 that $\mathfrak{sp}(4)$ $\mathfrak{sl}(4,\mathbb{R})$ and $\mathfrak{so}(3,1)$ cannot be extended to semi-simple not simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$, since the space of matrices that commute with them is of dimension two. Again from Proposition 2.2.3 in the case of $\mathfrak{sl}(3,\mathbb{R})$ the only way to have a semi-simple not simple representation where one summand is $\mathfrak{sl}(3,\mathbb{R})$, is to have a block diagonal representation of $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$ both in their definition representations.

For $\mathfrak{so}(3)$, it can be represented irreducibly in $\mathfrak{gl}(3,\mathbb{R})$, or $\mathfrak{gl}(5,\mathbb{R})$. In the first case, the only possibility for a semi-simple not simple subalgebra is to take a block diagonal sum of the definition representation of $\mathfrak{so}(3)$ with $\mathfrak{sl}(2,\mathbb{R})$. In the second case, where $\mathfrak{so}(3)$ is represented irreducibly in $\mathfrak{gl}(5,\mathbb{R})$, no extension is possible other than adding multiples of the identity by virtue of Proposition 2.2.1.

The four-dimensional representation of $\mathfrak{so}(3)$ must be reducible or else we have a contradiction to Schur's Lemma. In fact as is very well known, the Lie algebra $\mathfrak{so}(4) \approx \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ but the only way to obtain a semi-simple subalgebra of $\mathfrak{gl}(5,\mathbb{R})$ in which one summand is $\mathfrak{so}(3)$, is, in view of Proposition 2.2.3, to add an extra zero row and zero column to $\mathfrak{so}(4)$ since $\mathfrak{so}(4)$ is irreducible.

It remains to discuss the cases where the semi-simple subalgebra contains at least one copy of $\mathfrak{sl}(2,\mathbb{R})$, in each of its six inequivalent representations and in view of the discussion in this Section we need only look for subalgebras that are direct sums of copies of $\mathfrak{sl}(2,\mathbb{R})$. Looking at cases (a-f), Propositions 2.2.1 and 2.2.3 show that we cannot obtain a semi-simple not simple subalgebra in cases (a) and (b), respectively. Again in case (c) by Proposition 2.2.3 we can only obtain a block diagonal direct sum of type (c) and type (d). Again by Proposition 2.2.3 for type (d) the only new case we obtain is a block diagonal direct sum of 2×2 $\mathfrak{sl}(2,\mathbb{R})$ with itself augmented by an extra row and column of zeros.

In case (e) the only possibility is $\begin{bmatrix} aI_2 & bI_2 & 0 \\ cI_2 & dI_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

In case (f) the only matrix that commutes with E_1, E_2, E_3 is of the form $\begin{bmatrix} dI_3 & 0 \\ 0 & eI_2 \end{bmatrix}$: indeed note that putting $a = 1, b = c = 0$ gives a diagonal matrix with distinct eigenvalues and so the only matrix that commutes with it is another diagonal matrix.

Altogether we obtain six cases of semisimple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ that are not simple.

3.4 Semisimple algebras of $\mathfrak{gl}(5,\mathbb{R})$ and their representations

3.4.1 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

$$XVIII \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & f & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$XIX \quad \begin{bmatrix} a+d & b & e & 0 & 0 \\ c & d-a & 0 & e & 0 \\ f & 0 & a-d & b & 0 \\ 0 & f & c & -a-d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$XX \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 2d & 2e & 0 \\ 0 & 0 & f & 0 & e \\ 0 & 0 & 0 & 2f & -2d \end{bmatrix}$$

3.4.2 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$

$$XXI \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d & e & f \\ 0 & 0 & g & h & i \\ 0 & 0 & j & k & -(d+h) \end{bmatrix}$$

3.4.3 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)$

$$XXII \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & d & -e \\ 0 & 0 & -d & 0 & d \\ 0 & 0 & e & -d & 0 \end{bmatrix}$$

3.4.4 $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \approx \mathfrak{so}(4)$

$$XXIII \quad \begin{bmatrix} 0 & a & b & c & 0 \\ -a & 0 & d & e & 0 \\ -b & -d & 0 & f & 0 \\ -c & -e & -f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.5 Invariant Subspaces

3.5.1 Simple subalgebras

3.5.1.1 Simple subalgebras with an irreducible representation

There are 17 cases of simple subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ taking into account the different representations. We may apply Proposition 2.2.1 to the following six representations: $\mathfrak{sl}(2,\mathbb{R})I$, $\mathfrak{so}(3)IX$, $\mathfrak{sl}(5,\mathbb{R})XIV$, $\mathfrak{so}(5)XV$, $\mathfrak{so}(4,1)XVI$, $\mathfrak{so}(3,2)XVII$. As such there is nothing more to be said about these particular cases which give a reductive representation with a one-dimensional center.

3.5.1.2 Simple subalgebras with an irreducible 4×4 representation

The following five representations $\mathfrak{sl}(2,\mathbb{R})II$, $\mathfrak{so}(3)VIII$, $\mathfrak{sl}(4,\mathbb{R})XI$, $\mathfrak{so}(3,1)XII$, $\mathfrak{sp}(4)XIII$ consist of an irreducible 4×4 representation augmented by an extra row and column of zeros. Let us denote the 4×4 representation as A. Then A acts on $\mathfrak{gl}(5,\mathbb{R})$ by commutator and among the invariant subspaces of $\mathfrak{gl}(5,\mathbb{R})$, each of which is minimal, are:

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ y^t & 0 \end{bmatrix} \quad \begin{bmatrix} \lambda I_4 & 0 \\ 0 & \mu \end{bmatrix}$$

where $x, y \in \mathbb{R}^4$ and $\lambda, \mu \in \mathbb{R}$ and the last matrix comprises the kernel of the representation.

Now we examine each of the simple subalgebras that have an irreducible 4×4 representation and give an invariant complement to the space of invariant subspaces that have just been given.

3.5.1.3 $\mathfrak{sl}(2,\mathbb{R})II$

We have to add matrices of the form $\begin{bmatrix} -2e-j & -(2f+3k) & b & c & 0 \\ d & 3e & f & g & 0 \\ h & i & 3j & k & 0 \\ m & n & -(2i+3d) & -e-2j & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3.5.1.4 $\mathfrak{so}(3)VIII$

We have to add matrices of the form

$$\begin{bmatrix} 0 & d & e & f \\ -d & 0 & -f & e \\ -e & f & 0 & -d \\ -f & -e & d & 0 \end{bmatrix} \quad \begin{bmatrix} g & h & i & j \\ h & k & m & n \\ i & m & p & q \\ j & n & q & -(g+k+q) \end{bmatrix}.$$

3.5.1.5 $\mathfrak{sl}(4,\mathbb{R})XI$

The description given above furnishes all the minimal irreducible invariant subspaces.

3.5.1.6 $\mathfrak{so}(3,1)XII$

We have to add matrices of the form $\begin{bmatrix} a & b & c & d & 0 \\ b & e & f & g & 0 \\ c & h & i & j & 0 \\ -d & -g & -j & -(a+e+i) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3.5.1.7 $\mathfrak{sp}(4)XIII$

We have to add matrices of the form $\begin{bmatrix} d & e & 0 & g & 0 \\ f & -d & -g & 0 & 0 \\ 0 & h & d & f & 0 \\ -h & 0 & e & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let us now consider the possible invariant subspaces for the algebras in each of Subsubsections 3.5.1.5-3.5.1.4. Concerning $\mathfrak{sl}(4,\mathbb{R})$ in 3.5.1.5 according to Lemma 2.2.1 it is not possible to possible to have an invariant subspace in which x and y are both non-zero. Either we take $x = y = 0$, in which case we obtain subspaces of dimension 16 or 17 by taking at least one λ and μ not zero. We may take $\lambda = \mu = 0$ and $x \neq 0$ giving subalgebras of dimension 19. This latter subalgebra may itself be extended by taking at least one of λ and μ not zero, giving subalgebras of dimension 20 and 21. If instead we assume that $x = 0$ but that $y \neq 0$ we will only obtain a companion algebra of a subalgebra already obtained.

In Subsubsections 3.5.1.7-3.5.1.4 the additional invariant subspace is irreducible (check!) and provides a complement of the given subalgebra to $\mathfrak{sl}(4, \mathbb{R})$. Hence the extra invariant subspace cannot be included or else the semi-simple part of the sought after subalgebra will be $\mathfrak{sl}(4, \mathbb{R})$ and not $\mathfrak{so}(3, 1)$, $\mathfrak{sp}(4)$ or $\mathfrak{sl}(2, \mathbb{R})b$. As such, the same argument that was used for $\mathfrak{sl}(4, \mathbb{R})$ can be applied to $\mathfrak{so}(3, 1)$, $\mathfrak{sp}(4)$ or $\mathfrak{sl}(2, \mathbb{R})b$, resulting in solvable of extensions of dimensions one, two, four, five or six.

In view of the these last two subsections there remain six cases of simple subalgebra representations to consider.

3.5.1.8 $\mathfrak{sl}(3, \mathbb{R})$

We find that the invariant subspaces are as follows:

$$\begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & -(a+e) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q & r & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & u & v & 0 & 0 \end{bmatrix} \begin{bmatrix} w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & \alpha \end{bmatrix}.$$

In order to construct a radical we can proceed as follows. Suppose first of all that the lower left 3×2 and lower right 2×2 blocks are zero. We obtain a solvable subalgebra by letting the fourth partial column be arbitrary and the fifth partial column be zero or vice-versa but the latter is a companion of the former. We can also let both such partial column be arbitrary.

Again suppose that the lower left 3×2 block is zero. Suppose also that each of the partial columns in the fourth and fifth columns are both arbitrary. The lower right hand 2×2 block can now be any solvable subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ which are given by 2.1.

$$\begin{bmatrix} \epsilon x & x \\ 0 & \epsilon x \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & \lambda x \end{bmatrix} \begin{bmatrix} \lambda x & x \\ -x & \lambda x \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} (\lambda+1)x & y \\ 0 & \lambda x \end{bmatrix} \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}.$$

The next question is, suppose again that the lower left 3×2 matrix is zero. If the lower right hand 2×2 block can now be any solvable subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ is it possible to have just one of the partial column fourth and fifth columns arbitrary and the other non-zero? The answer is that for the four solvable upper triangular subalgebras of $\mathfrak{sl}(2, \mathbb{R})$ it is impossible to have the partial fourth and fifth columns proportional but otherwise arbitrary. If not both partial columns are arbitrary then it can only be that the fourth column is zero and the fifth arbitrary. In the two cases where the solvable subalgebras of $\mathfrak{sl}(2, \mathbb{R})$ involve complex quantities, both columns must be arbitrary. In case where the solvable subalgebra is one-dimensional and diagonal both columns must be arbitrary unless $\lambda = 1$ in which case it is possible to have the columns proportional. Finally in the two-dimensional diagonal solvable subalgebra case it is possible to have arbitrary fourth or fifth column but the columns cannot be proportional.

Next we note that it is possible to have an invariant subspace by combining a partial fifth column with a partial fourth row or vice-versa. We cannot combine a partial fifth column with a partial fifth row or partial fourth column with a partial fourth row else we will change the simple factor in the Levi decomposition. Furthermore we cannot "couple" the partial fifth column and partial fourth row as can be done when $\mathfrak{sl}(2, \mathbb{R})$ acts on $\mathfrak{gl}(4, \mathbb{R})$, for example.

3.5.1.9 $\mathfrak{so}(3)VII$

We find the invariant subspace are as follows:

$$\begin{bmatrix} 0 & b & c & 0 & 0 \\ -b & 0 & -a & 0 & 0 \\ -c & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ j & k & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ n & p & q & 0 & 0 \end{bmatrix} \begin{bmatrix} r & s & t & 0 & 0 \\ s & u & v & 0 & 0 \\ t & v & w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & y & z \\ 0 & 0 & 0 & z & \alpha \end{bmatrix}.$$

3.5.1.10 $\mathfrak{sl}(2, \mathbb{R}) III$

We find that the invariant subspace of $\mathfrak{sl}(2, \mathbb{R})$ are as follows:

$$\begin{bmatrix} 2a & 2b & 0 & 0 & 0 \\ c & 0 & b & 0 & 0 \\ 0 & 2c & -2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d & -2e & f & 0 & 0 \\ g & -2d & e & 0 & 0 \\ h & -2g & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q & r & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & u & v & 0 & 0 \end{bmatrix} \begin{bmatrix} w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & \alpha \end{bmatrix}.$$

3.5.1.11 $\mathfrak{sl}(2, \mathbb{R}) IV$

We find that the invariant subspaces of $\mathfrak{sl}(2, \mathbb{R})$ are as follows:

$$\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 \\ 0 & 0 & h & i & j \\ 0 & 0 & k & m & n \\ 0 & 0 & p & q & s \end{bmatrix} \begin{bmatrix} 0 & 0 & d & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ j & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ p & q & 0 & 0 & 0 \end{bmatrix}$$

Consider the commutator of two matrices

$$\begin{bmatrix} 0 & 0 & d & f & h \\ 0 & 0 & e & g & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ j & k & 0 & 0 & 0 \\ m & n & 0 & 0 & 0 \\ p & q & 0 & 0 & 0 \end{bmatrix}$$

which is given by

$$\begin{bmatrix} dj+fm+hp & dk+fn+hq & 0 & 0 & 0 \\ ej+gm+ip & ek+gn+iq & 0 & 0 & 0 \\ 0 & 0 & -dj-ek & -jf-kg & -jh-ki \\ 0 & 0 & -md-ne & -fm-gn & -mh-ni \\ 0 & 0 & -pd-qe & -pf-qg & -hp-iq \end{bmatrix}.$$

The upper left 2×2 block of this commutator can be arbitrary but the lower right 3×3 block must take values in a solvable subalgebra of $\mathfrak{gl}(3, \mathbb{R})$, all of which have been found in Section 2.1 above. Suppose first of all the commutator is upper triangular: then

$$md + ne = 0, pd + qe = 0, pf + qg = 0.$$

If d and e are arbitrary we must have that $m = n = p = q = 0$. Now the reduced (where $m = n = p = q = 0$) upper left 2×2 block must be a multiple of I_2, which gives $j = k = 0$. In that case we are back to the cases already discussed where the lower left 3×2 block is zero and the lower right 3×3 block gives a solvable subalgebra of $\mathfrak{gl}(3, \mathbb{R})$.

The next issue that we must address is happens when the subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ involves complex blocks. In this case it is not possible to have the third partial column zero and the fourth column non-zero or vice versa.

Now we assume that $d = e = 0$ and that f and g are arbitrary. Then if the lower right 3×3 block commutator is upper triangular we find that $p = q = 0$. Now if the upper left 2×2 block is a multiple of I_2 then $m = n = 0$. At this point we can construct several new subalgebras of $\mathfrak{gl}(5,\mathbb{R})$. We shall have different cases according as the partial columns with entries f, g and h, i are linearly dependent or not. In the case where they are dependent we can obtain subalgebras of the following form where any or all of m, q, s could be zero but all of a, b, c, f, g, j, k, n must be arbitrary. The algebras vary between dimensions of 8 and 11:

$$\begin{bmatrix} a & b & 0 & f & 0 \\ c & -a & 0 & g & 0 \\ j & k & m & n & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}.$$

In the case where the partial columns are linearly independent

$$\begin{bmatrix} a & b & 0 & f & h \\ c & -a & 0 & g & i \\ j & k & m & n & p \\ 0 & 0 & 0 & q & r \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

where $a, b, c, f, g, h, i, j, k, n, p$ must be arbitrary and there are various subcases depending on the values of m, q, r and s. The algebras vary between dimensions of 11 and 15.

The next issue that we must address is happens when the subalgebra of $\mathfrak{gl}(3,\mathbb{R})$ involves complex blocks. In this case it is not possible to have the third partial column zero and the fourth column non-zero or vice versa. Looking at 3.5.1.11 we see that

$$dj + ek = fm + gn, md + ne = -jf - kg.$$

If d, e are arbitrary it follows that $m = \lambda j, n = \lambda k$. Now if j, k are arbitrary 3.5.1.11 gives $d = \lambda f, e = \lambda g, \lambda d = -f, \lambda e = -g$ which gives $d = e = 0$ contrary to hypothesis. Thus if d, e are arbitrary the lower left block must be zero. The same argument applies if f, g are arbitrary. Thus assume $d = e = f = g = 0$. Then we must have $p = q = 0$.

There is also the possibility of "correlating" two invariant subspaces. Looking at $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.4}$

$$\begin{bmatrix} a & b & 0 & d-g & g \\ c & -a & 0 & e+h & -h \\ e & -d & 0 & -2f-i & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.5.1.12 $\mathfrak{sl}(2,\mathbb{R})V$

The invariant subspaces are given by:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} aI_2 & bI_2 & 0 \\ cI_2 & dI_2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x^t & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y^t & 0 \end{bmatrix}$$

where each of $A, B, C, D \in \mathfrak{sl}(2,\mathbb{R})$ and a, b, c, d, α correspond to the kernel of the representation and u, v, x, y are column vectors in $\mathbb{R}^2$.

To construct a solvable complement we can use, allowing for companion subalgebras, combinations of the second, sixth and seventh subspaces together with entries from the kernel furnishing variously complements of dimension

3.5.1.13 $\mathfrak{sl}(2,\mathbb{R})VI$

The invariant subspaces can be described as:

$$\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2d & 2e & 0 \\ 0 & 0 & f & 0 & e \\ 0 & 0 & 0 & 2f & -2d \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & -2h & i \\ 0 & 0 & j & -2g & h \\ 0 & 0 & k & -2j & g \end{bmatrix} \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & m & n & 0 \\ 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & p & -2q & r \\ 0 & 0 & s & -2p & q \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 \\ u & t & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ v & x & 0 & 0 & 0 \\ -2w & -2v & 0 & 0 & 0 \\ y & w & 0 & 0 & 0 \end{bmatrix}.$$

The only possibility for combining subspaces is to add the fifth and sixth and seventh and eighth as well as adding elements from the kernel.

3.5.2 Semi-simple Subalgebras

3.5.2.1 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})XVIII$

The invariant subspaces can be described as:

$$\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & f & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & g & i & 0 \\ 0 & 0 & h & j & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ q&r&0&0&0\\ s&t&0&0&0\\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ u&v&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&w&x&0 \end{bmatrix} \begin{bmatrix} y&0&0&0&0\\ 0&y&0&0&0\\ 0&0&z&0&0\\ 0&0&0&z&0\\ 0&0&0&0&\alpha \end{bmatrix}.$$

In addition to the first two invariant subspaces we can use any or all of the third, fourth and fifth subspaces so as to obtain subalgebras of dimensions $8, 10, 12$ and 14. Furthermore, each of these subalgebras may be modified by adding elements from the kernel increasing each of the subspaces by dimensions one, two and three.

3.5.2.2 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})XX$

The invariant subspaces can be described as:

$$\begin{bmatrix} 2a&2b&0&0&0\\ c&0&b&0&0\\ 0&2c&-2a&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} d&-2e&f&0&0\\ g&-2d&e&0&0\\ h&-2g&d&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&i&j\\ 0&0&0&k&-i \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&m&n\\ 0&0&0&p&q\\ 0&0&0&r&s\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ t&u&v&0&0\\ w&x&y&0&0 \end{bmatrix} \quad \begin{bmatrix} \lambda&0&0&0&0\\ 0&\lambda&0&0&0\\ 0&0&\lambda&0&0\\ 0&0&0&\mu&0\\ 0&0&0&0&\mu \end{bmatrix}.$$

To construct a solvable complement we can use only the second row of matrices. We put $B = \left[\begin{smallmatrix} m&n\\ p&q\\ r&s \end{smallmatrix}\right]$ $C = \left[\begin{smallmatrix} t&u&v\\ w&x&y \end{smallmatrix}\right]$. We must have $BC = \lambda I_3$ and $CB = \mu I_2$. Since B and C are not square we can only have $\lambda = 0$ and so $0 = BCB = \mu B$. Hence either $\mu = 0$ or $B = 0$ but if $B = 0$ then $\mu = 0$. So in conclusion $\lambda = \mu = 0$ and $BC = 0$ and $CB = 0$. Since $\mathfrak{sl}(2,\mathbb{R})$ in the given representation acts transitively on the subspace corresponding to the matrix

B if $B \neq 0$ we can only have that either $B = 0$ or $C = 0$, the latter giving the companion algebra of the former.

3.5.2.3 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) XIX$

The third representation of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ is given by $\begin{bmatrix} A+dI_2 & eI_2 & 0 \\ fI_2 & A-dI_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The 4×4 representation is irreducible and the possible extensions are similar to the case of $\mathfrak{sl}(4,\mathbb{R})$.

3.5.2.4 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) XXI$

The invariant subspaces can be described as:

$$\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & e & f \\ 0 & 0 & g & h & i \\ 0 & 0 & j & k & -(d+h) \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & m & n & p \\ 0 & 0 & q & r & s \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & u & 0 & 0 & 0 \\ v & w & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{bmatrix}.$$

It follows that we can have Levi subalgebras of dimensions $12, 13, 17, 18$ and 19 and similar companion subalgebras.

3.5.2.5 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3) XXII$

The invariant subspaces can be described as:

$$\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & -d & 0 & f \\ 0 & 0 & -e & -f & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & h & i \\ 0 & 0 & h & j & k \\ 0 & 0 & i & k & -(g+j) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & m & n & p \\ 0 & 0 & q & r & s \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & u & 0 & 0 & 0 \\ v & w & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{bmatrix}.$$

The existence of Levi subalgebndras follows the same pattern as in case XXI giving subalgebras of dimensions $7, 11, 12, 13$ and 14 and similar companion subalgebras.

3.5.2.6 $\mathfrak{so}(4)XXIII$

In fact $\mathfrak{so}(4)$ acts irreducibly om $\mathbb{R}^4$ so the analysis is similar to the case $\mathfrak{sl}(4, \mathbb{R})$: we obtain Levi subalgebras of dimension $7, 8, 10, 11$ and 12.

Chapter 4

Subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ with a non-trivial Levi decomposition by dimension

The final Chapter of this dissertation provides a list of all the subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ that have a non-trivial Levi decomposition. In what follows λ, μ, ν denote fixed but arbitrary sets of values and ϵ can only assume the values 0 or 1. Letters $a, b, c, d, ...$ are coordinates on the subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ that are given case by case.

4.0.3 Dimension 4

4.0.3.1 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$

$$I \quad \begin{bmatrix} 4a+d & 4b & 0 & 0 & 0 \\ c & 2a+d & 3b & 0 & 0 \\ 0 & 2c & d & 2b & 0 \\ 0 & 0 & 3c & d-2a & 0 \\ 0 & 0 & 0 & 4c & d-4a \end{bmatrix}$$

$$(\lambda^2+\mu^2\neq 0)\ \ II\begin{bmatrix} 3a+\lambda d & 3b & 0 & 0 & 0\\ c & a+\lambda d & 2b & 0 & 0\\ 0 & 2c & \lambda d-a & b & 0\\ 0 & 0 & 3c & \lambda d-3a & 0\\ 0 & 0 & 0 & 0 & \mu d\end{bmatrix}$$

$$(\lambda^2+\mu^2\neq 0)\quad III\begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 & 0\\ c & \lambda d & b & 0 & 0\\ 0 & 2c & \lambda d-2a & 0 & 0\\ 0 & 0 & 0 & \mu d & 0\\ 0 & 0 & 0 & 0 & \nu d\end{bmatrix} III\begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 & 0\\ c & \lambda d & b & 0 & 0\\ 0 & 2c & \lambda d-2a & 0 & 0\\ 0 & 0 & 0 & \epsilon d & d\\ 0 & 0 & 0 & 0 & \epsilon d\end{bmatrix}$$

$$III\begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 & 0\\ c & \lambda d & b & 0 & 0\\ 0 & 2c & \lambda d-2a & 0 & 0\\ 0 & 0 & 0 & \mu d & d\\ 0 & 0 & 0 & -d & \mu d\end{bmatrix} IV\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0\\ c & \lambda d-a & 0 & 0 & 0\\ 0 & 0 & \mu d & 0 & 0\\ 0 & 0 & 0 & \nu d & d\\ 0 & 0 & 0 & -d & \nu d\end{bmatrix}$$

$$(\epsilon=0,\lambda=0,1\,\mathrm{or}\,\epsilon=1)\,IV\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0\\ c & \lambda d-a & 0 & 0 & 0\\ 0 & 0 & \epsilon d & d & 0\\ 0 & 0 & 0 & \epsilon d & d\\ 0 & 0 & 0 & 0 & \epsilon d\end{bmatrix}$$

$$IV\ \ (\epsilon=0,\mu=0,1:\epsilon=\mu=0,\lambda=0,1)\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0\\ c & \lambda d-a & 0 & 0 & 0\\ 0 & 0 & \epsilon d & d & 0\\ 0 & 0 & 0 & \epsilon d & 0\\ 0 & 0 & 0 & 0 & \mu d\end{bmatrix}$$

$$(\lambda^2+\mu^2+\nu^2+\rho^2 \neq 0)\, IV \begin{bmatrix} a+\lambda d & b & 0 & 0 & 0 \\ c & \lambda d - a & 0 & 0 & 0 \\ 0 & 0 & \mu d & 0 & 0 \\ 0 & 0 & 0 & \nu d & 0 \\ 0 & 0 & 0 & 0 & \rho d \end{bmatrix}$$

4.0.3.2 $\lambda^2+\mu^2+\nu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V \begin{bmatrix} \lambda d + a & b & 0 & 0 & 0 \\ c & \lambda d - a & 0 & 0 & 0 \\ 0 & 0 & \mu d + a & b & 0 \\ 0 & 0 & c & \mu d - a & 0 \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.3.3 $\lambda^2+\mu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V \begin{bmatrix} \lambda d + a & b & d & 0 & 0 \\ c & \lambda d - a & 0 & d & 0 \\ 0 & 0 & \lambda d + a & b & 0 \\ 0 & 0 & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.3.4 $\lambda^2+\mu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V \begin{bmatrix} \lambda d + a & b & d & 0 & 0 \\ c & \lambda d - a & 0 & d & 0 \\ -d & 0 & \lambda d + a & b & 0 \\ 0 & -d & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.3.5 $\mathbb{R} \oplus \mathfrak{so}(3)$

$$VII\,(\lambda^2+\mu^2+\nu^2\neq 0)\begin{bmatrix} \lambda d & c & -b & 0 & 0 \\ -c & \lambda d & a & 0 & 0 \\ b & -a & \lambda d & 0 & 0 \\ 0 & 0 & 0 & \mu d & 0 \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix} VII \begin{bmatrix} \lambda d & c & -b & 0 & 0 \\ -c & \lambda d & a & 0 & 0 \\ b & -a & \lambda d & 0 & 0 \\ 0 & 0 & 0 & \epsilon d & d \\ 0 & 0 & 0 & 0 & \epsilon d \end{bmatrix}$$

$$VII \begin{bmatrix} \lambda d & c & -b & 0 & 0 \\ -c & \lambda d & a & 0 & 0 \\ b & -a & \lambda d & 0 & 0 \\ 0 & 0 & 0 & \mu d & d \\ 0 & 0 & 0 & -d & \mu d \end{bmatrix}$$

$$VIII\,(\lambda^2+\mu^2+\nu^2\neq 0)\begin{bmatrix} \lambda d & \mu d - a & -b & -c & 0 \\ a-\mu d & \lambda d & -c & b & 0 \\ b & c & \lambda d & -a-\mu d & 0 \\ c & -b & a+\mu d & \lambda d & 0 \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

$$IX \begin{bmatrix} d & -2\,c & -2\,b & -2\,a & 0 \\ 2\,c & d & 2\,a & -2\,b & 0 \\ \frac{b}{2} & -\frac{a}{2} & d & -c & 3\,a \\ \frac{a}{2} & \frac{b}{2} & c & d & 3\,b \\ 0 & 0 & -a & -b & d \end{bmatrix}$$

4.0.4 Dimension 5

4.0.4.1 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2$

$$II \begin{bmatrix} 3a+d & 3b & 0 & 0 & 0 \\ c & a+d & 2b & 0 & 0 \\ 0 & 2c & d-a & b & 0 \\ 0 & 0 & 3c & d-3a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.4.2 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1}$

$$III \begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 & 0 \\ c & \lambda d & b & 0 & 0 \\ 0 & 2c & \lambda d-2a & 0 & 0 \\ 0 & 0 & 0 & (\mu+1)d & e \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.4.3 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2$

$$III \begin{bmatrix} 2a+\lambda d+\mu e & 2b & 0 & 0 & 0 \\ c & \lambda d+\mu e & b & 0 & 0 \\ 0 & 2c & \lambda d+\mu e-2a & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$III \begin{bmatrix} 2a+\lambda d & 2b & 0 & 0 & 0 \\ c & \lambda d & b & 0 & 0 \\ 0 & 2c & \lambda d-2a & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & d \end{bmatrix} III \begin{bmatrix} 2a+\lambda d+\mu e & 2b & 0 & 0 & 0 \\ c & \lambda d+\mu e & b & 0 & 0 \\ 0 & 2c & \lambda d+\mu e-2a & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & -e & d \end{bmatrix}$$

$$III \begin{bmatrix} 2a+d & 2b & 0 & 0 & 0 \\ c & d & b & 0 & 0 \\ 0 & 2c & d-2a & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & \lambda e \end{bmatrix} III \begin{bmatrix} 2a+d & 2b & 0 & 0 & 0 \\ c & d & b & 0 & 0 \\ 0 & 2c & d-2a & 0 & 0 \\ 0 & 0 & 0 & \epsilon e & e \\ 0 & 0 & 0 & 0 & \epsilon e \end{bmatrix}$$

$$III \begin{bmatrix} 2a+d & 2b & 0 & 0 & 0 \\ c & d & b & 0 & 0 \\ 0 & 2c & d-2a & 0 & 0 \\ 0 & 0 & 0 & \lambda e & e \\ 0 & 0 & 0 & -e & \lambda e \end{bmatrix}$$

$$IV \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & \rho d+\sigma e \end{bmatrix}$$

$$IV \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & \epsilon d+\nu e \end{bmatrix}$$

$$IV \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & \epsilon d & d & 0 \\ 0 & 0 & 0 & \epsilon d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$IV \quad \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & \epsilon d & d & \nu e \\ 0 & 0 & 0 & \epsilon d+\epsilon e & 0 \\ 0 & 0 & 0 & \delta e & \epsilon d+\epsilon e \end{bmatrix}$$

$$IV \quad \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & -e & d & 0 \\ 0 & 0 & 0 & 0 & \sigma d+\nu e \end{bmatrix}$$

$$IV \quad \begin{bmatrix} a+\lambda d+\mu e & b & 0 & 0 & 0 \\ c & \lambda d+\mu e-a & 0 & 0 & 0 \\ 0 & 0 & \delta d & e & 0 \\ 0 & 0 & -e & \delta d & 0 \\ 0 & 0 & 0 & 0 & \sigma d+\nu e \end{bmatrix}$$

4.0.4.4 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1}$

$$IV \quad \begin{bmatrix} a+\lambda d & b & 0 & 0 & 0 \\ c & \lambda d-a & 0 & 0 & 0 \\ 0 & 0 & \mu d & e & 0 \\ 0 & 0 & d & (\mu+1)d & d \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

$$IV\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0 \\ c & \lambda d - a & 0 & 0 & 0 \\ 0 & 0 & \mu d & e & d \\ 0 & 0 & 0 & (\mu+1)d & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix} IV\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0 \\ c & \lambda d - a & 0 & 0 & 0 \\ 0 & 0 & \mu d & e & 0 \\ 0 & 0 & 0 & (\mu+1)d & 0 \\ 0 & 0 & 0 & 0 & (\mu+1)d \end{bmatrix}$$

$$IV\begin{bmatrix} a+\lambda d & b & 0 & 0 & 0 \\ c & \lambda d - a & 0 & 0 & 0 \\ 0 & 0 & \mu d & e & \mu d \\ 0 & 0 & 0 & (\mu+1)d & e \\ 0 & 0 & 0 & 0 & (\mu+2)d \end{bmatrix}$$

4.0.4.5 $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2 = A_{5.40}$

$$IV\begin{bmatrix} a & b & d & 0 & 0 \\ c & -a & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.4.6 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.3}$

$$V\begin{bmatrix} d+a & b & 0 & 0 & 0 \\ c & d-a & 0 & 0 & 0 \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d + \mu e \end{bmatrix}$$

4.0.4.7 $\mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V \begin{bmatrix} d+a & b & e & 0 & 0 \\ c & d-a & 0 & e & 0 \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d \end{bmatrix}$$

4.0.4.8 $\lambda^2 + \mu^2 \neq 0 \, \mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V \begin{bmatrix} \lambda d+a & b & 0 & 0 & 0 \\ c & \lambda d-a & 0 & 0 & 0 \\ 0 & 0 & \mu d+a & b & 0 \\ 0 & 0 & c & \mu d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix} V \begin{bmatrix} \lambda d+a & b & d & 0 & 0 \\ c & \lambda d-a & 0 & d & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$V \begin{bmatrix} \lambda d+a & b & d & 0 & 0 \\ c & \lambda d-a & 0 & d & 0 \\ -d & 0 & \lambda d+a & b & 0 \\ 0 & -d & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.4.9 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1}$

$$V \begin{bmatrix} (\lambda+1)d+a & b & e & 0 & 0 \\ c & (\lambda+1)d-a & 0 & e & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.4.10 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2$

$$V \begin{bmatrix} a & b & 0 & 0 & d \\ c & -a & 0 & 0 & e \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.4.11 $\mathfrak{so}(3) \oplus \mathbb{R}^2$

$$VII \begin{bmatrix} \lambda d + \mu e & c & -b & 0 & 0 \\ -c & \lambda d + \mu e & a & 0 & 0 \\ b & -a & \lambda d + \mu e & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix} VII \begin{bmatrix} \lambda d & c & -b & 0 & 0 \\ -c & \lambda d & a & 0 & 0 \\ b & -a & \lambda d & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

$$VII \begin{bmatrix} \lambda d + \mu e & c & -b & 0 & 0 \\ -c & \lambda d + \mu e & a & 0 & 0 \\ b & -a & \lambda d + \mu e & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & -e & d \end{bmatrix}$$

4.0.4.12 $\mathfrak{so}(3) \oplus A_{2.1}$

$$VII \begin{bmatrix} \lambda d & c & -b & 0 & 0 \\ -c & \lambda d & a & 0 & 0 \\ b & -a & \lambda d & 0 & 0 \\ 0 & 0 & 0 & (\mu+1)d & e \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.4.13 $\mathfrak{so}(3) \oplus \mathbb{R}^2$

$$VIII \begin{bmatrix} \lambda d & \mu d - a & -b & -c & 0 \\ a - \mu d & \lambda d & -c & b & 0 \\ b & c & \lambda d & -a - \mu d & 0 \\ c & -b & a + \mu d & \lambda d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$VIII \begin{bmatrix} d & \mu e - a & -b & -c & 0 \\ a - \mu e & \lambda d & -c & b & 0 \\ b & c & \lambda d & -a - \mu e & 0 \\ c & -b & a + \mu e & \lambda d & 0 \\ 0 & 0 & 0 & 0 & \nu d + \rho e \end{bmatrix}$$

4.0.5 Dimension 6

4.0.5.1 $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^3$

$$III \begin{bmatrix} 2c & 2a & 0 & d & 0 \\ b & 0 & a & e & 0 \\ 0 & 2b & -2c & f & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

4.0.5.2 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1})$

$$III \begin{bmatrix} \lambda d + \mu f + 2c & 2a & 0 & 0 & 0 \\ b & \lambda d + \mu f & a & 0 & 0 \\ 0 & 2b & \lambda d + \mu f - 2c & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & f \end{bmatrix},$$

4.0.5.3 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{2.1})$

$$IV \begin{bmatrix} a & b & d & 0 & 0 \\ c & -a & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda f & 0 \\ 0 & 0 & 0 & 0 & \mu f \end{bmatrix} IV \begin{bmatrix} a & b & d & 0 & 0 \\ c & -a & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon f & f \\ 0 & 0 & 0 & 0 & \epsilon f \end{bmatrix} IV \begin{bmatrix} a & b & d & 0 & 0 \\ c & -a & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda f & f \\ 0 & 0 & 0 & -f & \lambda f \end{bmatrix}$$

4.0.5.4 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.1}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + \beta e & d & f \\ 0 & 0 & 0 & \alpha d + \beta e & e \\ 0 & 0 & 0 & 0 & \alpha d + \beta e \end{bmatrix}.$$

4.0.5.5 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.2}$

$$IV\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & (\alpha+1)f & f & d \\ 0 & 0 & 0 & (\alpha+1)f & e \\ 0 & 0 & 0 & 0 & \alpha f \end{bmatrix}$$

4.0.5.6 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.3,3.4,3.5}$

$$IV\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & (\beta+1)f & 0 & d \\ 0 & 0 & 0 & (a+\beta)f & e \\ 0 & 0 & 0 & 0 & \beta f \end{bmatrix}.$$

4.0.5.7 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.6,3.7}$

$$IV\begin{bmatrix} d & b & 0 & 0 & 0 \\ c & -d & 0 & 0 & 0 \\ 0 & 0 & (a+\beta)f & f & d \\ 0 & 0 & -f & (a+\beta)f & e \\ 0 & 0 & 0 & 0 & \beta f \end{bmatrix}.$$

4.0.5.8 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3$

$$IV\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix} IV\begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & f & 0 & d \\ 0 & 0 & 0 & f & e \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.5.9 $\mathfrak{sl}(2,\mathbb{R}) \oplus (A_{2.1} \oplus \mathbb{R})$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + \lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d + \lambda f & 0 \\ 0 & 0 & 0 & 0 & \beta d + \mu f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + \lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d + \lambda f & 0 \\ 0 & 0 & 0 & 0 & \alpha d + \mu f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + \lambda f & e & d + \mu f \\ 0 & 0 & 0 & (\alpha+1)d + \lambda f & 0 \\ 0 & 0 & 0 & 0 & \alpha d + \lambda f \end{bmatrix} IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & 0 & d + f & e \\ 0 & 0 & 0 & 0 & d + 2f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + \lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d + \lambda f & 0 \\ 0 & 0 & 0 & 0 & (\alpha+1)d + \mu f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d+\lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d+\lambda f & 0 \\ 0 & 0 & 0 & d+\mu f & (\alpha+1)d+\lambda f \end{bmatrix}$$

4.0.5.10 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.1} = L_{6.2}$

$$IV \begin{bmatrix} a & b & 0 & d & 0 \\ c & -a & 0 & e & 0 \\ e & -d & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.5.11 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3$

$$V \begin{bmatrix} a & b & e & 0 & 0 \\ c & d & 0 & e & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix} V \begin{bmatrix} d+a & b & 0 & 0 & 0 \\ c & d-a & 0 & 0 & 0 \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix} V \begin{bmatrix} a & b & e & 0 & 0 \\ c & d & 0 & e & 0 \\ -e & 0 & a & b & 0 \\ 0 & -e & c & d & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.5.12 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{2.1})$

$$V \begin{bmatrix} d+a & b & f & 0 & 0 \\ c & d-a & 0 & f & 0 \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

$$V\begin{bmatrix} (\lambda+1)d+a & b & e & 0 & 0 \\ c & (\lambda+1)d-a & 0 & e & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.5.13 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3$

$$V\begin{bmatrix} a & b & d & e & 0 \\ c & -a & f & -d & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.5.14 $\lambda^2+\mu^2+\nu^2 \neq 0 \, \mathbb{R}^3 \oplus \mathfrak{sl}(2,\mathbb{R})$

$$V\begin{bmatrix} \lambda d+a & b & 0 & 0 & e \\ c & \lambda d-a & 0 & 0 & f \\ 0 & 0 & \mu d+a & b & 0 \\ 0 & 0 & c & \mu d-a & 0 \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.5.15 $\lambda-\mu \neq 0 \, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3}$

$$V\begin{bmatrix} \lambda d+a & b & d & 0 & e \\ c & \lambda d-a & 0 & d & f \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.5.16 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3$

$$V \begin{bmatrix} d+a & b & d & 0 & e \\ c & d-a & 0 & d & f \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.5.17 $\mathfrak{so}(3) \rtimes \mathbb{R}^3 = L_{6.1}$

$$VII \begin{bmatrix} 0 & c & -b & d & 0 \\ -c & 0 & a & e & 0 \\ b & -a & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.5.18 $\mathbb{R}^3 \oplus \mathfrak{so}(3)$

$$VIII \begin{bmatrix} d & e & -b & -c & 0 \\ -e & d & -c & b & 0 \\ b & c & d & -a & 0 \\ c & -b & a & d & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.6 Dimension 7

4.0.6.1 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4$

$$II \begin{bmatrix} 3a & 3b & 0 & 0 & d \\ c & a & 2b & 0 & e \\ 0 & 2c & -a & b & f \\ 0 & 0 & 3c & -3a & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.6.2 $\epsilon = 1, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{4.5a=b=1}; \epsilon = 0, \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$III \begin{bmatrix} 2c & 2a & 0 & 0 & d \\ b & 0 & a & 0 & e \\ 0 & 2b & -2c & 0 & f \\ 0 & 0 & 0 & \epsilon g & g \\ 0 & 0 & 0 & 0 & \epsilon g \end{bmatrix},$$

4.0.6.3 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$III \begin{bmatrix} 2c & 2a & 0 & d & 0 \\ b & 0 & a & e & 0 \\ 0 & 2b & -2c & f & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & \lambda g \end{bmatrix},$$

4.0.6.4 $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.6,3.7}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d+(a+\beta)g & g & e \\ 0 & 0 & -g & d+(a+\beta)g & f \\ 0 & 0 & 0 & 0 & d+\beta g \end{bmatrix}$$

4.0.6.5 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{4.8,4.9b}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & (\alpha+b+1)d & e & f \\ 0 & 0 & 0 & (\alpha+b)d & g \\ 0 & 0 & 0 & 0 & \alpha d \end{bmatrix}.$$

4.0.6.6 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{4.12}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \lambda g+\mu d & d & e \\ 0 & 0 & -d & \lambda g+\mu d & f \\ 0 & 0 & 0 & 0 & (\lambda+1)g+\mu d \end{bmatrix}.$$

4.0.6.7 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2 \oplus A_{2.1}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix} IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \lambda e + g & f & d \\ 0 & 0 & 0 & (\lambda+1)e + g & 0 \\ 0 & 0 & 0 & 0 & \lambda e + g \end{bmatrix}.$$

4.0.6.8 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1} \oplus A_{2.1}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \lambda e + \mu g & f & d \\ 0 & 0 & 0 & (\lambda+1)e + \mu g & 0 \\ 0 & 0 & 0 & 0 & \lambda e + \nu g \end{bmatrix}$$

4.0.6.9 $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.1}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d + \lambda e + \mu f & e & g \\ 0 & 0 & 0 & d + \lambda e + \mu f & f \\ 0 & 0 & 0 & 0 & d + \lambda e + \mu f \end{bmatrix}$$

4.0.6.10 $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.2}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d+(\lambda+1)g & g & e \\ 0 & 0 & 0 & d+(\lambda+1)g & f \\ 0 & 0 & 0 & 0 & d+\lambda g \end{bmatrix}.$$

4.0.6.11 $\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus A_{3.3,3.4,3.5}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d+(\beta+1)g & 0 & e \\ 0 & 0 & 0 & d+(\alpha+\beta)g & f \\ 0 & 0 & 0 & 0 & d+\beta g \end{bmatrix}.$$

4.0.6.12 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3})$

$$V \begin{bmatrix} d+a & b & f & 0 & 0 \\ c & d-a & 0 & f & 0 \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix}$$

4.0.6.13 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$V \begin{bmatrix} a & b & d & e & 0 \\ c & -a & f & g & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.6.14 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4$

$$V \begin{bmatrix} a & b & 0 & 0 & d \\ c & -a & 0 & 0 & e \\ 0 & 0 & a & b & f \\ 0 & 0 & c & -a & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.6.15 $\lambda^2 + \mu^2 \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4$

$$V \begin{bmatrix} \lambda d + a & b & e & f & 0 \\ c & \lambda d - a & g & -e & 0 \\ 0 & 0 & \lambda d + a & b & 0 \\ 0 & 0 & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.6.16 $\lambda \neq \mu\, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{4.5a=b=1}$

$$V\begin{bmatrix} \lambda d + a & b & e & f & 0 \\ c & \lambda d - a & g & -e & 0 \\ 0 & 0 & \mu d + a & b & 0 \\ 0 & 0 & c & \mu d - a & 0 \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.6.17 $\lambda^2 + \mu^2 \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4$

$$V\begin{bmatrix} \lambda d + a & b & d + e & f & 0 \\ c & \lambda d - a & g & d - e & 0 \\ 0 & 0 & \lambda d + a & b & 0 \\ 0 & 0 & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.6.18 $\lambda^2 + \mu^2 \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4$

$$V\begin{bmatrix} \lambda d + a & b & d + e & f & 0 \\ c & \lambda d - a & g & d - e & 0 \\ -d & 0 & \lambda d + a & b & 0 \\ 0 & -d & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.6.19 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$V \begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.6.20 $(\lambda-1)^2+\mu^2 \neq 0 \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3})$

$$V \begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.6.21 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & d+\mu e \end{bmatrix}$$

4.0.6.22 $\lambda \neq 1 \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3}$

$$V\begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.6.23 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3})$

$$V\begin{bmatrix} \lambda d+a & b & 0 & 0 & f \\ c & \lambda d-a & 0 & 0 & g \\ 0 & 0 & \mu d+a & b & 0 \\ 0 & 0 & c & \mu d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.6.24 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3a=b=1})$

$$V\begin{bmatrix} \lambda d+a & b & \mu d & 0 & f \\ c & \lambda d-a & 0 & \mu d & g \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.6.25 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{4.5a=b=\lambda-\mu+1}$

$$V\begin{bmatrix} (\lambda+1)d+a & b & e & 0 & f \\ c & (\lambda+1)d-a & 0 & e & g \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.6.26 $\mathfrak{so}(3) \rtimes A_{4.5a=b=1}$

$$VII\begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ 0 & 0 & 0 & g & g \\ 0 & 0 & 0 & 0 & g \end{bmatrix} \quad VII\begin{bmatrix} 0 & c & -b & d & 0 \\ -c & 0 & a & e & 0 \\ b & -a & 0 & f & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & \lambda g \end{bmatrix}.$$

4.0.6.27 $\mathbb{R} \oplus (\mathfrak{so}(3) \rtimes \mathbb{R}^3)$

$$VII\begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.6.28 $\mathfrak{so}(3) \rtimes \mathbb{R}^4$

$$VIII \begin{bmatrix} 0 & a & b & c & d \\ -a & 0 & c & -b & e \\ -b & -c & 0 & a & f \\ -c & b & -a & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.6.29 $(\lambda^2 + \mu^2 \neq 0)\, \mathbb{R} \oplus \mathfrak{so}(3,1)$

$$XII \begin{bmatrix} \lambda g & a & b & c & 0 \\ -a & \lambda g & d & e & 0 \\ -b & -d & \lambda g & f & 0 \\ c & e & f & \lambda g & 0 \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}.$$

4.0.6.30 $\lambda^2 + \mu^2 + \nu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})$

$$XVIII \begin{bmatrix} a + \lambda g & b & 0 & 0 & 0 \\ c & \lambda g - a & 0 & 0 & 0 \\ 0 & 0 & d + \mu g & e & 0 \\ 0 & 0 & f & \mu g - d & 0 \\ 0 & 0 & 0 & 0 & \nu g \end{bmatrix}.$$

4.0.6.31 $\lambda^2+\mu^2\neq 0\,\mathbb{R}\oplus\mathfrak{sl}(2,\mathbb{R})\oplus(\mathfrak{sl}(2,\mathbb{R})$

$$XIX\begin{bmatrix} \lambda g+d+a & b & e & 0 & 0\\ c & \lambda g+d-a & 0 & e & 0\\ f & 0 & \lambda g+a-d & b & 0\\ 0 & f & c & \lambda g-a-d & 0\\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}.$$

4.0.6.32 $\mathbb{R}\oplus(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))$

$$XX\begin{bmatrix} a+\lambda g & b & 0 & 0 & 0\\ c & +\lambda g-a & 0 & 0 & 0\\ 0 & 0 & 2d+\mu g & 2e & 0\\ 0 & 0 & f & \mu g & e\\ 0 & 0 & 0 & 2f & +\mu g-2d \end{bmatrix}$$

4.0.6.33 $\lambda^2+\mu^2\neq 0\,\mathbb{R}\oplus(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(3))$

$$XXII\begin{bmatrix} a+\lambda g & b & 0 & 0 & 0\\ c & -a+\lambda g & 0 & 0 & 0\\ 0 & 0 & \mu g & d & -e\\ 0 & 0 & -d & \mu g & f\\ 0 & 0 & e & -f & \mu g \end{bmatrix}$$

4.0.6.34 $\lambda^2 + \mu^2 \neq 0 \, \mathbb{R} \oplus \mathfrak{so}(4)$

$$XXIII \begin{bmatrix} \lambda g & a & b & c & 0 \\ -a & \lambda g & d & e & 0 \\ -b & -d & \lambda g & f & 0 \\ -c & -e & -f & \lambda g & f \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}$$

4.0.7 Dimension 8

4.0.7.1 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4)$

$$II \begin{bmatrix} h+3a & 3b & 0 & 0 & d \\ c & h+a & 2b & 0 & e \\ 0 & 2c & h-a & b & f \\ 0 & 0 & 3c & h-3a & g \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

4.0.7.2 $\lambda^2 + \mu^2 \neq 0 \, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7(a=b=c=\lambda-\mu)}$

$$II \begin{bmatrix} \lambda h+3a & 3b & 0 & 0 & d \\ c & \lambda h+a & 2b & 0 & e \\ 0 & 2c & \lambda h-a & b & f \\ 0 & 0 & 3c & \lambda h-3a & g \\ 0 & 0 & 0 & 0 & \mu h \end{bmatrix}$$

4.0.7.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=c=\lambda-\mu}$

$$III \begin{bmatrix} \lambda g+2c & 2a & 0 & 0 & d \\ b & \lambda g & a & 0 & e \\ 0 & 2b & \lambda g-2c & 0 & f \\ 0 & 0 & 0 & (1+\mu)g & h \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix},$$

4.0.7.4 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{4.5a=b=1})$

$$III \begin{bmatrix} \lambda g+\mu h+2c & 2a & 0 & d & 0 \\ b & \lambda g+\mu h & a & e & 0 \\ 0 & 2b & \lambda g+\mu h-2c & f & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix},$$

4.0.7.5 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} \oplus A_{4.5a=b=1}$

$$III \begin{bmatrix} \lambda g+2c & 2a & 0 & 0 & d \\ b & \lambda g & a & 0 & e \\ 0 & 2b & \lambda g-2c & 0 & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & g \end{bmatrix},$$

4.0.7.6 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3))$

$$IV \quad \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & f & g \\ 0 & 0 & -f & e & h \\ 0 & 0 & -g & -h & e \end{bmatrix}$$

4.0.7.7 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{5.36}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & (\alpha+1)e+\beta d & f & h \\ 0 & 0 & 0 & \alpha e+(\beta+1)d & g \\ 0 & 0 & 0 & 0 & \alpha e+\beta q \end{bmatrix}.$$

4.0.7.8 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} \oplus A_{4.8,4.9}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & d+(\beta+1)e & f & h \\ 0 & 0 & 0 & d+\beta e & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.7.9 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} \oplus A_{4.12}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & h & e & f \\ 0 & 0 & -e & h & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.7.10 $\mathfrak{sl}(2,\mathbb{R}) \oplus A_{2.1} \oplus A_{3.3}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & \alpha d + f & e & h \\ 0 & 0 & 0 & (\alpha+1)d + f & 0 \\ 0 & 0 & 0 & 0 & \alpha q + g \end{bmatrix}.$$

4.0.7.11 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R} \oplus A_{2.1} \oplus A_{2.1}$

$$IV \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & -a & 0 & 0 & 0 \\ 0 & 0 & f & d & e \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

4.0.7.12 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{3.3})$

$$V\begin{bmatrix} a & b & e & 0 & g \\ c & d & 0 & e & h \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix} V\begin{bmatrix} d+a & b & 0 & 0 & g \\ c & d-a & 0 & 0 & h \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.7.13 $(\lambda+\mu-1)\mu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{2.1} \oplus A_{3.3})$

$$V\begin{bmatrix} d+a & b & f & 0 & g \\ c & d-a & 0 & f & h \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

$$V\begin{bmatrix} (\lambda+1)d+a & b & f & 0 & g \\ c & (\lambda+1)d-a & 0 & f & h \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.7.14 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{4.5a=b=1})$

$$V\begin{bmatrix} d+a & b & f & g & 0 \\ c & d-a & h & -f & 0 \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.7.15 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$V\begin{bmatrix} d+a & b & e+f & g & 0 \\ c & d-a & h & e-f & 0 \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.7.16 $\lambda - \mu \neq 0\, \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{4.5a=b=1})$

$$V\begin{bmatrix} \lambda d+a & b & f & g & 0 \\ c & \lambda d-a & h & -f & 0 \\ 0 & 0 & \mu d+a & b & 0 \\ 0 & 0 & c & \mu d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.7.17 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4)$

$$V\begin{bmatrix} \lambda d+a & b & d+f & g & 0 \\ c & \lambda d-a & h & d-f & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.7.18 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=c=1}$

$$V\begin{bmatrix} (\lambda+1)d+a & b & e & f & 0 \\ c & (\lambda+1)d-a & g & h & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.7.19 $\lambda - \nu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=\frac{\mu-\nu}{\lambda-\nu}}$

$$V\begin{bmatrix} \lambda d+a & b & 0 & 0 & e \\ c & \lambda d-a & 0 & 0 & f \\ 0 & 0 & \mu d+a & b & g \\ 0 & 0 & c & \mu d-a & h \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.7.20 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4)$

$$V\begin{bmatrix} d+a & b & 0 & 0 & e \\ c & d-a & 0 & 0 & f \\ 0 & 0 & d+a & b & g \\ 0 & 0 & c & d-a & h \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.7.21 $\lambda - \mu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \ltimes A_{5.15a=1}$

$$V \begin{bmatrix} \lambda d + a & b & d & 0 & e \\ c & \lambda d - a & 0 & d & f \\ 0 & 0 & \lambda d + a & b & g \\ 0 & 0 & c & \lambda d - a & h \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.7.22 $\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{5.1}$

$$V \begin{bmatrix} d + a & b & d & 0 & e \\ c & d - a & 0 & d & f \\ 0 & 0 & d + a & b & g \\ 0 & 0 & c & d - a & h \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.7.23 $\lambda^2 + \mu^2 \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \ltimes A_{5.17p=q=\lambda-\mu s=1}$

$$V \begin{bmatrix} \lambda d + a & b & d & 0 & e \\ c & \lambda d - a & 0 & d & f \\ -d & 0 & \lambda d + a & b & g \\ 0 & -d & c & \lambda d - a & h \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.7.24 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^5$

$$V \begin{bmatrix} a & b & d & e & g \\ c & -a & f & -d & h \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & -a & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.7.25 $\mathfrak{so}(3) \rtimes A_{5.7a=b=c=-\lambda}$

$$VII \begin{bmatrix} \lambda h & c & -b & 0 & d \\ -c & \lambda h & a & 0 & e \\ b & -a & \lambda h & 0 & f \\ 0 & 0 & 0 & (1+\mu)h & g \\ 0 & 0 & 0 & 0 & \mu h \end{bmatrix}$$

4.0.7.26 $\mathfrak{so}(3) \oplus \mathbb{R} \oplus A_{4.5a=b=1}$

$$VII \begin{bmatrix} \lambda h & c & -b & 0 & d \\ -c & \lambda h & a & 0 & e \\ b & -a & \lambda h & 0 & f \\ 0 & 0 & 0 & h & g \\ 0 & 0 & 0 & 0 & h \end{bmatrix} \quad VII \begin{bmatrix} \lambda g+\mu h & c & -b & d & 0 \\ -c & \lambda g+\mu h & a & e & 0 \\ b & -a & \lambda g+\mu h & f & 0 \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

4.0.7.27 $\lambda^2+\mu^2+\nu^2 \neq 0\, \mathfrak{so}(3) \rtimes A_{5.17 p=q=\frac{\lambda-\nu}{\mu}, s=1}$

$$VIII \begin{bmatrix} \lambda d & \mu d - a & -b & -c & e \\ a-\mu d & \lambda d & -c & b & f \\ b & c & \lambda d & -a-\mu d & g \\ c & -b & a+\mu d & \lambda d & h \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.7.28 $\mathbb{R}^2 \oplus \mathfrak{so}(3,1)$

$$XII \begin{bmatrix} g & a & b & c & 0 \\ -a & g & d & e & 0 \\ -b & -d & g & f & 0 \\ c & e & f & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.7.29 $\mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})$

$$XVIII \begin{bmatrix} a+g & b & 0 & 0 & 0 \\ c & g-a & 0 & 0 & 0 \\ 0 & 0 & d+h & e & 0 \\ 0 & 0 & f & h-d & 0 \\ 0 & 0 & 0 & 0 & \lambda g + \mu h \end{bmatrix}.$$

4.0.7.30 $\mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})$

$$XVIII \begin{bmatrix} a+\lambda g & b & 0 & 0 & 0 \\ c & \lambda g - a & 0 & 0 & 0 \\ 0 & 0 & d+\mu g & e & 0 \\ 0 & 0 & f & \mu g - d & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.7.31 $\mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2)$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & g \\ c & -a & 0 & 0 & h \\ 0 & 0 & d & e & 0 \\ 0 & 0 & f & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.7.32 $\lambda^2 + \mu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})$

$$XIX \begin{bmatrix} g+d+a & b & e & 0 & 0 \\ c & g+d-a & 0 & e & 0 \\ f & 0 & g+a-d & b & 0 \\ 0 & f & c & g-a-d & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.7.33 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$

$$XX \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 2e+h & 2f & 0 \\ 0 & 0 & g & h & f \\ 0 & 0 & 0 & 2g & h-2e \end{bmatrix}$$

4.0.7.34 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3))$

$$XXII \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & f & g \\ 0 & 0 & -f & e & h \\ 0 & 0 & -g & -h & e \end{bmatrix}$$

4.0.7.35 $\mathbb{R}^2 \oplus \mathfrak{so}(4)$

$$XXIII \begin{bmatrix} g & a & b & c & 0 \\ -a & g & d & e & 0 \\ -b & -d & g & f & 0 \\ -c & -e & -f & g & f \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

4.0.8 Dimension 9

4.0.8.1 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=c=1})$

$$II \begin{bmatrix} h+3a & 3b & 0 & 0 & d \\ c & h+a & 2b & 0 & e \\ 0 & 2c & h-a & b & f \\ 0 & 0 & 3c & h-3a & g \\ 0 & 0 & 0 & 0 & i \end{bmatrix}$$

4.0.8.2 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{2.1} \oplus A_{4.5a=b=1})$

$$III \begin{bmatrix} \lambda g+\mu i+2c & 2a & 0 & 0 & d \\ b & \lambda g+\mu i & a & 0 & e \\ 0 & 2b & \lambda g+\mu i-2c & 0 & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & i \end{bmatrix}$$

4.0.8.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^6$

$$III \begin{bmatrix} 2c & 2a & 0 & d & g \\ b & 0 & a & e & h \\ 0 & 2b & -2c & f & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.8.4 $\mathfrak{sl}(2,\mathbb{R}) \oplus N_{(6.13,\beta=\delta=a+1,\alpha=\gamma=b)}$

$$IV\begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & \alpha g+\beta d & d & e \\ 0 & 0 & -d & \alpha g+\beta d & f \\ 0 & 0 & 0 & 0 & (\alpha+1)g+\beta d \end{bmatrix}.$$

4.0.8.5 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.7})$

$$IV\begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & d+(\alpha+\beta)g & g & e \\ 0 & 0 & -g & d+(\alpha+\beta)g & f \\ 0 & 0 & 0 & 0 & d+\beta g \end{bmatrix}.$$

4.0.8.6 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes A_{5.8})$

$$IV\begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & (\alpha+\beta+1)d & e & g \\ 0 & 0 & 0 & (\alpha+\beta)d & f \\ 0 & 0 & 0 & 0 & \alpha d \end{bmatrix}.$$

4.0.8.7 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes (\mathbb{R}^2 \oplus A_{3.1}))$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & d & e & g \\ 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.8.8 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes A_{4.8}) = \mathfrak{sl}(2,\mathbb{R}) \oplus N_{6.2,\alpha=1,\beta=\gamma=0}$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & d+(\alpha+1)g & g & e \\ 0 & 0 & 0 & d+(\alpha+1)g & f \\ 0 & 0 & 0 & 0 & d+\alpha g \end{bmatrix}.$$

4.0.8.9 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.1} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & d+\alpha e+\beta f & e & g \\ 0 & 0 & 0 & d+\alpha e+\beta f & f \\ 0 & 0 & 0 & 0 & d+\alpha e+\beta f \end{bmatrix}.$$

4.0.8.10 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \oplus A_{2.1} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & \alpha e + \lambda g & f & d \\ 0 & 0 & 0 & (\alpha+1)e + \lambda g & 0 \\ 0 & 0 & 0 & 0 & \alpha e + \mu g \end{bmatrix} \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & \alpha e + g & f & d \\ 0 & 0 & 0 & (\alpha+1)e + g & 0 \\ 0 & 0 & 0 & 0 & \alpha e + g \end{bmatrix}$$

4.0.8.11 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{2.1} \oplus A_{2.1} \oplus A_{2.1})$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix}.$$

4.0.8.12 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3} \oplus A_{3.5})$

$$IV \begin{bmatrix} a & b & 0 & 0 & h \\ c & -a & 0 & 0 & i \\ 0 & 0 & d + (\beta+1)g & 0 & e \\ 0 & 0 & 0 & d + (\alpha+\beta)g & f \\ 0 & 0 & 0 & 0 & d + \beta f \end{bmatrix}.$$

4.0.8.13 $\mathbb{R}^3 \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3)$

$$V\begin{bmatrix} a & b & e & f & 0 \\ c & d & g & h & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & i \end{bmatrix}$$

4.0.8.14 $\mathbb{R}^3 \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^3)$

$$V\begin{bmatrix} d+a & b & g & h & 0 \\ c & d-a & i & -g & 0 \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.8.15 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{4.5})$

$$V\begin{bmatrix} d+a & b & f & g & 0 \\ c & d-a & h & i & 0 \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

$$V\begin{bmatrix} (\lambda+1)d+a & b & f & g & 0 \\ c & (\lambda+1)d-a & h & i & 0 \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.16 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.3})$

$$V\begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & h \\ 0 & 0 & c & e-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.17 $\lambda \neq 1 \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.54 a=1 b=0}$

$$V\begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.18 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.54 a=1 b=0}$

$$V\begin{bmatrix} \lambda d+a & b & d & 0 & f \\ c & \lambda d-a & 0 & d & g \\ 0 & 0 & \lambda d+a & b & h \\ 0 & 0 & c & \lambda d-a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.19 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.17s=1p=q=\mu})$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & d+\mu e \end{bmatrix}$$

4.0.8.20 $\lambda \neq 1\, \mathfrak{sl}(2,\mathbb{R}) \rtimes N_{6.18}$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ -e & 0 & d+a & b & h \\ 0 & -e & c & d-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.21 $\mu \neq 1\, \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.15a=1})$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ -e & 0 & d+a & b & h \\ 0 & -e & c & d-a & i \\ 0 & 0 & 0 & 0 & d+\mu e \end{bmatrix}$$

4.0.8.22 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.1})$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ -e & 0 & d+a & b & h \\ 0 & -e & c & d-a & i \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.8.23 $\lambda - \mu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{3.3})$

$$V \begin{bmatrix} \lambda d+a & b & 0 & 0 & f \\ c & \lambda d-a & 0 & 0 & g \\ 0 & 0 & \mu d+a & b & h \\ 0 & 0 & c & \mu d-a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.24 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=c=1})$

$$V \begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.25 $\mathfrak{sl}(2,\mathbb{R}) \rtimes N_{6.18}$

$$V\begin{bmatrix} \lambda d+a & b & d & 0 & f \\ c & \lambda d-a & 0 & d & g \\ -d & 0 & \lambda d+a & b & h \\ 0 & -d & c & \lambda d-a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.26 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{2.1} \oplus A_{3.3}))$

$$V\begin{bmatrix} d+a & b & f & 0 & h \\ c & d-a & 0 & f & i \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix}$$

4.0.8.27 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$V\begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & h \\ 0 & 0 & c & e-a & i \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.8.28 $\lambda + \mu = 1\,\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^6$

$$V\begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & h \\ 0 & 0 & c & e-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.29 $\lambda + \mu \neq 1\,\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3a=1} \oplus A_{3.3a=1})$

$$V\begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & e+a & b & h \\ 0 & 0 & c & e-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.30 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.1})$

$$V\begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.8.31 $\mu \neq 0\, \mathbb{R} \oplus (\mathfrak{sl}(2, \mathbb{R}) \rtimes A_{5.15a=1})$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & d+\mu e \end{bmatrix}$$

4.0.8.32 $\lambda \neq 1 \mathbb{R} \oplus (\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R} \oplus A_{5.1}))$

$$V \begin{bmatrix} d+a & b & e & 0 & f \\ c & d-a & 0 & e & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.8.33 $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathbb{R}^6$

$$V \begin{bmatrix} d+a & b & 0 & 0 & f \\ c & d-a & 0 & 0 & g \\ 0 & 0 & d+a & b & h \\ 0 & 0 & c & d-a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.34 $\lambda \neq \mu\, \mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3a=1} \oplus A_{3.3a=1})$

$$V \begin{bmatrix} \lambda d + a & b & 0 & 0 & f \\ c & \lambda d - a & 0 & 0 & g \\ 0 & 0 & \mu d + a & b & h \\ 0 & 0 & c & \mu d - a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.35 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{5.7a=b=c=1})$

$$V \begin{bmatrix} \lambda d + a & b & 0 & 0 & f \\ c & \lambda d - a & 0 & 0 & g \\ 0 & 0 & \lambda d + a & b & h \\ 0 & 0 & c & \lambda d - a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.36 $\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{6.54a=1b=0}$

$$V \begin{bmatrix} \lambda d + a & b & d & 0 & f \\ c & \lambda d - a & 0 & d & g \\ 0 & 0 & \lambda d + a & b & h \\ 0 & 0 & c & \lambda d - a & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.8.37 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.53}$

$$\begin{array}{c}V\end{array}\begin{bmatrix}(\lambda+1)d+a & b & e & 0 & f\\ c & (\lambda+1)d-a & 0 & e & g\\ 0 & 0 & \lambda d+a & b & h\\ 0 & 0 & c & \lambda d-a & i\\ 0 & 0 & 0 & 0 & (\lambda+1)d\end{bmatrix}$$

4.0.8.38 $\lambda \neq \mu - 1\, \mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.54 a=1 b=\frac{1}{\lambda-\mu+1}}$

$$\begin{array}{c}V\end{array}\begin{bmatrix}(\lambda+1)d+a & b & e & 0 & f\\ c & (\lambda+1)d-a & 0 & e & g\\ 0 & 0 & \lambda d+a & b & h\\ 0 & 0 & c & \lambda d-a & i\\ 0 & 0 & 0 & 0 & \mu d\end{bmatrix}$$

4.0.8.39 $\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{6.82 \delta=\frac{\mu-\nu}{2}, a=b=\lambda-\frac{\mu+\nu}{2}}$

$$\begin{array}{c}VI\end{array}\begin{bmatrix}2a & 2b & 0 & d & e\\ c & 0 & b & f & g\\ 0 & 2c & -2a & h & i\\ 0 & 0 & 0 & a & b\\ 0 & 0 & 0 & c & -a\end{bmatrix}$$

4.0.8.40 $\mathfrak{so}(3) \rtimes \mathbb{R}^6$

$$VII \begin{bmatrix} 0 & c & -b & d & g \\ -c & 0 & a & e & h \\ b & -a & 0 & f & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.8.41 $\mathfrak{so}(3) \rtimes (A_{2.1} \oplus A_{4.5a=b=1})$

$$VII \begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ 0 & 0 & 0 & h & g \\ 0 & 0 & 0 & 0 & i \end{bmatrix}.$$

4.0.8.42 $\mathfrak{so}(3) \rtimes N_{6.18}$

$$VIII \begin{bmatrix} \lambda d & \mu d - a & -b & -c & f \\ a - \mu d & \lambda d & -c & b & g \\ b & c & \lambda d & -a - \mu d & h \\ c & -b & a + \mu d & \lambda d & i \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$VIII \begin{bmatrix} \lambda d & \mu e - a & -b & -c & f \\ a - \mu e & \lambda d & -c & b & g \\ b & c & \lambda d & -a - \mu e & h \\ c & -b & a + \mu e & \lambda d & i \\ 0 & 0 & 0 & 0 & \nu d + \rho e \end{bmatrix}$$

4.0.8.43 $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & f & 0 \\ 0 & 0 & g & h & 0 \\ 0 & 0 & 0 & 0 & i \end{bmatrix}$$

4.0.8.44 $\lambda \neq \nu\, \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{3.3})$

$$XVIII \begin{bmatrix} a+\lambda i & b & 0 & 0 & g \\ c & -a+\lambda i & 0 & 0 & h \\ 0 & 0 & d+\mu i & e & 0 \\ 0 & 0 & f & -d+\mu i & 0 \\ 0 & 0 & 0 & 0 & \nu i \end{bmatrix}$$

4.0.8.45 $\lambda^2+\mu^2 \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$XVIII \begin{bmatrix} a+\lambda i & b & 0 & 0 & g \\ c & -a+\lambda i & 0 & 0 & h \\ 0 & 0 & d+\mu i & e & 0 \\ 0 & 0 & f & -d+\mu i & 0 \\ 0 & 0 & 0 & 0 & \lambda i \end{bmatrix}$$

4.0.9 Dimension 10

4.0.9.1 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$III \begin{bmatrix} \lambda j + 2c & 2a & 0 & d & g \\ b & \lambda j & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & \mu j \end{bmatrix}$$

4.0.9.2 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$III \begin{bmatrix} \lambda j + 2c & 2a & 0 & d & g \\ b & \lambda j & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & j & j \\ 0 & 0 & 0 & 0 & j \end{bmatrix},$$

4.0.9.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes N^7$

$$III \begin{bmatrix} (\lambda) j + 2c & 2a & 0 & d & g \\ b & (\lambda j & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.9.4 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes \mathbb{R}^6)$

$$III \begin{bmatrix} \lambda j + 2c & 2a & 0 & d & g \\ b & \lambda j & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & \mu j & j \\ 0 & 0 & 0 & -j & \mu j \end{bmatrix}$$

4.0.9.5 $\mathfrak{sl}(2,\mathbb{R}) \ltimes H^7$

$$III \begin{bmatrix} 2c & 2a & 0 & 0 & d \\ b & 0 & a & 0 & e \\ 0 & 2b & -2c & 0 & f \\ h & i & j & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

4.0.9.6 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes \mathbb{R}^4)$

$$IV \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & h & e & f \\ 0 & 0 & -e & h & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}.$$

4.0.9.7 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes A_{5.1})$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & d & e & 0 \\ 0 & 0 & 0 & d+f & e \\ 0 & 0 & 0 & 0 & d+2f \end{bmatrix}.$$

4.0.9.8 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes \mathbb{R}^5)$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & \alpha d+\lambda f & e & d+\mu f \\ 0 & 0 & 0 & (\alpha+1)d+\lambda f & 0 \\ 0 & 0 & 0 & 0 & \alpha d+\mu f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & \alpha d+\lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d+\lambda f & 0 \\ 0 & 0 & 0 & 0 & \beta d+\mu f \end{bmatrix}$$

4.0.9.9 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{2.1} \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & f & 0 & d \\ 0 & 0 & 0 & f & e \\ 0 & 0 & 0 & 0 & f \end{bmatrix}.$$

4.0.9.10 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes (\mathbb{R} \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix} IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & (\beta+1)f & 0 & d \\ 0 & 0 & 0 & (\beta+1)f & e \\ 0 & 0 & 0 & 0 & \beta f \end{bmatrix}$$

$$IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & (\alpha+1)f & f & d \\ 0 & 0 & 0 & (\alpha+1)f & e \\ 0 & 0 & 0 & 0 & \alpha f \end{bmatrix} IV \begin{bmatrix} a & b & 0 & g & i \\ c & -a & 0 & h & j \\ 0 & 0 & \alpha d+\beta e & d & f \\ 0 & 0 & 0 & \alpha d+\beta e & e \\ 0 & 0 & 0 & 0 & \alpha d+\beta e \end{bmatrix}$$

4.0.9.11 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes (\mathbb{R}^2 \oplus A_{3.1}))$

$$IV \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & d+(\beta+1)e & f & h \\ 0 & 0 & 0 & d+\beta e & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.9.12 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{2.1} \oplus A_{2.1} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & f & d & e \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.9.13 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{2.1} \oplus A_{2.1} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & \alpha d + f & e & h \\ 0 & 0 & 0 & (1+\alpha)d + f & 0 \\ 0 & 0 & 0 & 0 & \alpha d + g \end{bmatrix}.$$

4.0.9.14 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes (\mathbb{R}^2 \oplus A_{3.1}))$

$$IV \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & (\alpha+1)e + \beta d & f & h \\ 0 & 0 & 0 & d + \alpha e + (\beta+1)d & g \\ 0 & 0 & 0 & 0 & d + \alpha e + \beta d \end{bmatrix}.$$

4.0.9.15 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes A_{5.1}))$

$$V \begin{bmatrix} a & b & f & 0 & g \\ c & d & 0 & f & h \\ 0 & 0 & a & b & i \\ 0 & 0 & c & d & j \\ 0 & 0 & 0 & 0 & e \end{bmatrix} V \begin{bmatrix} d+a & b & 0 & 0 & g \\ c & d-a & 0 & 0 & h \\ 0 & 0 & e+a & b & i \\ 0 & 0 & c & e-a & j \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.9.16 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes N_{6.18\alpha=\beta=\gamma=1})$

$$V \begin{bmatrix} a & b & e & 0 & g \\ c & d & 0 & e & h \\ -e & 0 & a & b & i \\ 0 & -e & c & d & j \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.9.17 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes A_{5.1})$

$$V \begin{bmatrix} d+a & b & f & 0 & g \\ c & d-a & 0 & f & h \\ 0 & 0 & a+e & b & i \\ 0 & 0 & c & e-a & j \\ 0 & 0 & 0 & 0 & \lambda d + \mu e \end{bmatrix}$$

$$V \begin{bmatrix} (\lambda+1)d+a & b & f & 0 & g \\ c & (\lambda+1)d-a & 0 & f & h \\ 0 & 0 & \lambda d + a & b & i \\ 0 & 0 & c & \lambda d - a & j \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.9.18 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes A_{2.1})$

$$V \begin{bmatrix} d+a & b & g & h & 0 \\ c & d-a & i & j & 0 \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.9.19 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes A_{5.7a=b=c=1})$

$$V \begin{bmatrix} d+a & b & g & h & 0 \\ c & d-a & i & j & 0 \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.9.20 $\lambda + \mu \neq 1\, \mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{4.5a=b=1})$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & -f & j \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d + \mu e \end{bmatrix}$$

4.0.9.21 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^5))$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & -f & j \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d + (1-\lambda) e \end{bmatrix}$$

4.0.9.22 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \oplus (\mathbb{R} \rtimes \mathbb{R}^4))$

$$V \begin{bmatrix} d+a & b & e & f & g \\ c & d-a & h & i & j \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d \end{bmatrix}$$

4.0.9.23 $\lambda - \mu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3} \oplus A_{4.5a=b=1})$

$$V \begin{bmatrix} \lambda d+a & b & f & g & h \\ c & \lambda d-a & i & -f & j \\ 0 & 0 & \mu d+a & b & 0 \\ 0 & 0 & c & \mu d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.9.24 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^4 \oplus A_{3.3})$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & -f & j \\ 0 & 0 & d+a & b & 0 \\ 0 & 0 & c & d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.9.25 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^4 \oplus A_{3.3})$

$$V\begin{bmatrix} \lambda d+a & b & d+f & g & h \\ c & \lambda d-a & i & d-f & j \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.9.26 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes \mathbb{R}^6)$

$$V\begin{bmatrix} (\lambda+1)d+a & b & e & f & g \\ c & (\lambda+1)d-a & h & i & j \\ 0 & 0 & \lambda d+a & b & 0 \\ 0 & 0 & c & \lambda d-a & 0 \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.9.27 $\mathfrak{sl}(2,\mathbb{R}) \ltimes K^7$

$$V\begin{bmatrix} a & b & d & e & g \\ c & -a & f & -d & h \\ 0 & 0 & a & b & i \\ 0 & 0 & c & -a & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.9.28 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$VI\begin{bmatrix} \lambda j+2a & 2b & 0 & d & e \\ c & \lambda j & b & f & g \\ 0 & 2c & \lambda j-2a & h & i \\ 0 & 0 & 0 & a+\mu j & b \\ 0 & 0 & 0 & c & \mu j-a \end{bmatrix}$$

4.0.9.29 $\mathfrak{so}(3) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$VII\begin{bmatrix} \lambda j & c & -b & d & g \\ -c & \lambda j & a & e & h \\ b & -a & \lambda j & f & i \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & \mu j \end{bmatrix}.$$

4.0.9.30 $\mathfrak{so}(3) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$VII\begin{bmatrix} \lambda j & c & -b & d & g \\ -c & \lambda j & a & e & h \\ b & -a & \lambda j & f & i \\ 0 & 0 & 0 & j & j \\ 0 & 0 & 0 & 0 & j \end{bmatrix}.$$

4.0.9.31 $\mathfrak{so}(3) \rtimes N^7$

$$VII \begin{bmatrix} \lambda j & c & -b & d & g \\ -c & \lambda j & a & e & h \\ b & -a & \lambda j & f & i \\ 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.9.32 $\mathfrak{so}(3) \rtimes H^7$

$$VII \begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ h & i & j & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.9.33 $\mathfrak{so}(3,1) \rtimes \mathbb{R}^4$

$$XII \begin{bmatrix} 0 & a & b & c & g \\ -a & 0 & d & e & h \\ -b & -d & 0 & f & i \\ c & e & f & 0 & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.9.34 $(\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2)$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & g \\ c & -a & 0 & 0 & h \\ 0 & 0 & d & e & i \\ 0 & 0 & f & -d & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.9.35 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes \mathbb{R}^4$

$$XVIII \begin{bmatrix} a & b & g & h & 0 \\ c & -a & i & j & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & f & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.9.36 $\mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2)$

$$XVIII \begin{bmatrix} a+g & b & 0 & 0 & i \\ c & g-a & 0 & 0 & j \\ 0 & 0 & d+h & e & 0 \\ 0 & 0 & f & h-d & 0 \\ 0 & 0 & 0 & 0 & \lambda g + \mu h \end{bmatrix}.$$

4.0.9.37 $\lambda \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})) \ltimes A_{3.3})$

$$XVIII \begin{bmatrix} a+\lambda g & b & 0 & 0 & i \\ c & \lambda g - a & 0 & 0 & j \\ 0 & 0 & d+\mu g & e & 0 \\ 0 & 0 & f & \mu g - d & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.9.38 $\mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^2)$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & i \\ c & -a & 0 & 0 & j \\ 0 & 0 & d+g & e & 0 \\ 0 & 0 & f & g-d & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.9.39 $(\mathfrak{sl}(2,\mathbb{R}) \oplus (\mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^4$

$$XIX \begin{bmatrix} d+a & b & e & 0 & g \\ c & d-a & 0 & e & h \\ f & 0 & a-d & b & i \\ 0 & f & c & -a-d & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.9.40 $\mathfrak{so}(4) \rtimes \mathbb{R}^4$

$$XXIII \begin{bmatrix} 0 & a & b & c & g \\ -a & 0 & d & e & h \\ -b & -d & 0 & f & i \\ -c & -e & -f & 0 & j \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.10 Dimension 11

4.0.10.1 $\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R}^6 \oplus A_{2.1})$

$$III \begin{bmatrix} \lambda j + 2c & 2a & 0 & d & g \\ b & \lambda j & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & (1+\mu)j & k \\ 0 & 0 & 0 & 0 & \mu j \end{bmatrix},$$

4.0.10.2 $\mathfrak{sl}(2, \mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$III \begin{bmatrix} \lambda j + \mu k + 2c & 2a & 0 & d & g \\ b & \lambda j + \mu k & a & e & h \\ 0 & 2b & \lambda j + \mu k - 2c & f & i \\ 0 & 0 & 0 & j & \nu k \\ 0 & 0 & 0 & -\nu k & j \end{bmatrix},$$

4.0.10.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$III \begin{bmatrix} \lambda j + 2c & 2a & 0 & d & g \\ b & \lambda j + & a & e & h \\ 0 & 2b & \lambda j - 2c & f & i \\ 0 & 0 & 0 & j & k \\ 0 & 0 & 0 & 0 & j \end{bmatrix},$$

4.0.10.4 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$III \begin{bmatrix} \lambda j + \mu k + 2c & 2a & 0 & d & g \\ b & \lambda j + \mu k + & a & e & h \\ 0 & 2b & \lambda j + \mu k + -2c & f & i \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix},$$

4.0.10.5 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes H^7)$

$$III \begin{bmatrix} 2c & 2a & 0 & 0 & d \\ b & 0 & a & 0 & e \\ 0 & 2b & -2c & 0 & f \\ h & i & j & \lambda k & g \\ 0 & 0 & 0 & 0 & \mu k \end{bmatrix},$$

4.0.10.6 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & (\alpha+\beta+1)d & e & g \\ 0 & 0 & 0 & (\alpha+\beta)d & f \\ 0 & 0 & 0 & 0 & \alpha d \end{bmatrix}.$$

4.0.10.7 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes (\mathbb{R} \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & d+(\alpha+1)g & g & e \\ 0 & 0 & 0 & d+(\alpha+1)g & f \\ 0 & 0 & 0 & 0 & d+d+\alpha g \end{bmatrix}.$$

4.0.10.8 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & d+\alpha e+\beta f & e & g \\ 0 & 0 & 0 & d+\alpha e+\beta f & f \\ 0 & 0 & 0 & 0 & d+\alpha e+\beta f \end{bmatrix}.$$

4.0.10.9 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & \alpha e + \lambda g & f & d \\ 0 & 0 & 0 & (\alpha+1)e + \lambda g & 0 \\ 0 & 0 & 0 & 0 & \alpha e + \mu g \end{bmatrix}.$$

4.0.10.10 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{2.1} \oplus A_{3.3} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix}.$$

4.0.10.11 $\mathfrak{sl}(2,\mathbb{R}) \oplus (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & \alpha e + g & f & d \\ 0 & 0 & 0 & (\alpha+1)e + g & 0 \\ 0 & 0 & 0 & 0 & \alpha e + g \end{bmatrix}.$$

4.0.10.12 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes (\mathbb{R}^2 \oplus A_{3.1}))$

$$IV \begin{bmatrix} a & b & 0 & 0 & j \\ c & -a & 0 & 0 & k \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & d \\ 0 & 0 & 0 & 0 & e \end{bmatrix}.$$

4.0.10.13 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes (\mathbb{R} \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & 0 & h & j \\ c & -a & 0 & i & k \\ 0 & 0 & d+(\beta+1)g & 0 & e \\ 0 & 0 & 0 & d+(\alpha+\beta)g & f \\ 0 & 0 & 0 & 0 & d+\beta f \end{bmatrix}.$$

4.0.10.14 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3} \oplus A_{4.5a=b=1}))$

$$V \begin{bmatrix} d+a & b & g & h & i \\ c & d-a & j & -g & k \\ 0 & 0 & e+a & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}$$

4.0.10.15 $\lambda + \mu \neq 1$ $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3} \oplus A_{5.7a=b=c=1})$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d + \mu e \end{bmatrix}$$

4.0.10.16 $\lambda \neq 1$ $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes \mathbb{R}^6))$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & \lambda d + (1-\lambda) e \end{bmatrix}$$

4.0.10.17 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \oplus A_{5.7a=b=c=1}))$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.10.18 $\mathbb{R} \oplus (\mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathbb{R}^4 \oplus A_{3.3}))$

$$V \begin{bmatrix} a & b & f & g & h \\ c & d & i & j & k \\ 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.10.19 $\mathbb{R} \oplus (\mathfrak{sl}(2, \mathbb{R}) \ltimes (A_{3.3} \oplus A_{4.5a=b=1}))$

$$V \begin{bmatrix} (\lambda+1)d + a & b & f & g & h \\ c & (\lambda+1)d - a & i & j & k \\ 0 & 0 & \lambda d + a & b & 0 \\ 0 & 0 & c & \lambda d - a & 0 \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.10.20 $\mathbb{R} \oplus (\mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes A_{5.1}))$

$$V \begin{bmatrix} d + a & b & f & 0 & h \\ c & d - a & 0 & f & i \\ 0 & 0 & a + e & b & j \\ 0 & 0 & c & e - a & k \\ 0 & 0 & 0 & 0 & g \end{bmatrix}$$

4.0.10.21 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{4.5a=b=1}))$

$$V \begin{bmatrix} a+d & b & f & g & i \\ c & d-a & h & -f & j \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix}$$

4.0.10.22 $\mathfrak{sl}(2,\mathbb{R}) \rtimes K^8$

$$V \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & a & b & j \\ 0 & 0 & c & -a & k \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.10.23 $(\lambda-\mu)^2 + (\lambda-\nu)^2 + (\mu-\nu)^2 \neq 0, \mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes K^7)$

$$V \begin{bmatrix} \lambda d+a & b & e & f & h \\ c & \lambda d-a & g & -e & i \\ 0 & 0 & \mu d+a & b & j \\ 0 & 0 & c & \mu d-a & k \\ 0 & 0 & 0 & 0 & \nu d \end{bmatrix}$$

4.0.10.24 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes K^7)$

$$V \begin{bmatrix} d+a & b & d+e & f & h \\ c & d-a & g & d-e & i \\ 0 & 0 & d+a & b & j \\ 0 & 0 & c & d-a & k \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

4.0.10.25 $\lambda - \mu \neq 0\, \mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes K^7)$

$$V \begin{bmatrix} \lambda d+a & b & d+e & f & h \\ c & \lambda d-a & g & d-e & i \\ 0 & 0 & \lambda d+a & b & j \\ 0 & 0 & c & \lambda d-a & k \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.10.26 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$VI \begin{bmatrix} j+2a & 2b & 0 & d & e \\ c & j & b & f & g \\ 0 & 2c & j-2a & h & i \\ 0 & 0 & 0 & a+k & b \\ 0 & 0 & 0 & c & k-a \end{bmatrix}$$

4.0.10.27 $\mathfrak{so}(3) \rtimes (\mathbb{R} \rtimes N^7)$

$$VII \begin{bmatrix} \lambda k & c & -b & d & g \\ -c & \lambda k & a & e & h \\ b & -a & \lambda k & f & i \\ 0 & 0 & 0 & (\mu+1)k & m \\ 0 & 0 & 0 & 0 & \mu k \end{bmatrix}$$

4.0.10.28 $\mathfrak{so}(3) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$VII \begin{bmatrix} \lambda k+\mu m & c & -b & d & g \\ -c & \lambda k+\mu m & a & e & h \\ b & -a & \lambda k+\mu m & f & i \\ 0 & 0 & 0 & k & m \\ 0 & 0 & 0 & -m & k \end{bmatrix}$$

4.0.10.29 $\mathfrak{so}(3) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$VII \begin{bmatrix} \lambda k+\mu m & c & -b & d & g \\ -c & \lambda k+\mu m & a & e & h \\ b & -a & \lambda k+\mu m & f & i \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.10.30 $\mathfrak{so}(3) \rtimes (\mathbb{R} \rtimes N^7)$

$$VII \begin{bmatrix} \lambda k & c & -b & d & g \\ -c & \lambda k & a & e & h \\ b & -a & \lambda k & f & i \\ 0 & 0 & 0 & k & m \\ 0 & 0 & 0 & 0 & k \end{bmatrix}$$

4.0.10.31 $\lambda^2 + \mu^2 \neq 0\, \mathfrak{so}(3) \rtimes (\mathbb{R} \rtimes H^7)$

$$VII \begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ h & i & j & \lambda k & g \\ 0 & 0 & 0 & 0 & \mu k \end{bmatrix}.$$

4.0.10.32 $\mathfrak{sl}(3, \mathbb{R}) \rtimes \mathbb{R}^3$

$$X \begin{bmatrix} a & b & c & i & 0 \\ d & e & f & j & 0 \\ g & h & -(a+e) & k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.10.33 $\mathbb{R} \oplus (\mathfrak{so}(3,1) \rtimes \mathbb{R}^4)$

$$XII \begin{bmatrix} g & a & b & c & h \\ -a & g & d & e & i \\ -b & -d & g & f & j \\ c & e & f & g & k \\ 0 & 0 & 0 & 0 & g \end{bmatrix}.$$

4.0.10.34 $\lambda \neq \mu(\mathfrak{so}(3,1) \rtimes A_{5.7a=b=c=1})$

$$XII \begin{bmatrix} \lambda g & a & b & c & h \\ -a & \lambda g & d & e & i \\ -b & -d & \lambda g & f & j \\ c & e & f & \lambda g & k \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}.$$

4.0.10.35 $\lambda^2 + \mu^2 \neq 0 \, \mathbb{R} \oplus \mathfrak{sp}(4)$

$$XIII \begin{bmatrix} a + \lambda k & b & e & f & 0 \\ c & d + \lambda k & f & g & 0 \\ h & i & \lambda k - a & -c & 0 \\ i & j & -b & \lambda k - d & 0 \\ 0 & 0 & 0 & 0 & \mu k \end{bmatrix}.$$

4.0.10.36 $\mathbb{R} \oplus \mathfrak{so}(5)$

$$XV \begin{bmatrix} k & a & b & c & g \\ -a & k & d & e & h \\ -b & -d & k & f & i \\ -c & -e & -f & k & j \\ -g & -h & -i & j & k \end{bmatrix}.$$

4.0.10.37 $\mathbb{R} \oplus \mathfrak{so}(4,1)$

$$XVI \begin{bmatrix} k & a & b & c & d \\ -a & k & d & e & h \\ -b & -d & k & f & i \\ -c & -e & -f & k & j \\ g & h & i & i & k \end{bmatrix}.$$

4.0.10.38 $\mathbb{R} \oplus \mathfrak{so}(3,2)$

$$XVII \begin{bmatrix} k & a & b & c & g \\ -a & k & d & e & h \\ -b & -d & k & f & i \\ c & e & f & k & j \\ g & h & i & j & k \end{bmatrix}.$$

4.0.10.39 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus ((\mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus L_{6.3}$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & i \\ c & d & 0 & 0 & j \\ 0 & 0 & e & f & 0 \\ 0 & 0 & g & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix}$$

4.0.10.40 $\lambda^2 + \mu^2 + \nu^2 \neq 0\, \mathbb{R} \oplus ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^4)$

$$XVIII \begin{bmatrix} a+\lambda k & b & g & h & 0 \\ c & \lambda k - a & i & j & 0 \\ 0 & 0 & d+\mu k & e & 0 \\ 0 & 0 & f & \mu k - d & 0 \\ 0 & 0 & 0 & 0 & \nu k \end{bmatrix}$$

4.0.10.41 $\lambda^2 + \mu^2 + \nu^2 \neq 0\, \mathbb{R} \oplus (((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^4)$

$$XVIII \begin{bmatrix} a+\lambda k & b & 0 & 0 & g \\ c & -a+\lambda k & 0 & 0 & h \\ 0 & 0 & d+\mu k & e & i \\ 0 & 0 & f & \mu k - d & j \\ 0 & 0 & 0 & 0 & \nu k \end{bmatrix}$$

4.0.10.42 $((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes A_{5.7a=b=c=1}$

$$XIX \begin{bmatrix} a+d & b & e & 0 & h \\ c & -a+d & 0 & e & i \\ f & 0 & a-d & b & j \\ 0 & f & c & -a-d & k \\ 0 & 0 & 0 & 0 & g \end{bmatrix}$$

4.0.10.43 $\lambda^2 + \mu^2 \neq 0\, \mathbb{R} \oplus ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes \mathbb{R}^4)$

$$XIX \begin{bmatrix} a+d+\lambda g & b & e & 0 & h \\ c & -a+d+\lambda g & 0 & e & i \\ f & 0 & a-d+\lambda g & b & j \\ 0 & f & c & -a-d+\lambda g & k \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}$$

4.0.10.44 $\mathfrak{so}(4) \rtimes A_{5.7a=b=c=\lambda-\mu}$

$$XXIII \begin{bmatrix} \lambda g & a & b & c & h \\ -a & \lambda g & d & e & i \\ -b & -d & \lambda g & f & j \\ -c & -e & -f & \lambda g & k \\ 0 & 0 & 0 & 0 & \mu g \end{bmatrix}$$

4.0.11 Dimension 12

4.0.11.1 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes N^7)$

$$III\begin{bmatrix} 2c & 2a & 0 & d & g \\ b & 0 & a & e & h \\ 0 & 2b & -2c & f & i \\ 0 & 0 & 0 & j & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.2 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes H^7)$

$$III\begin{bmatrix} 2c & 2a & 0 & 0 & d \\ b & 0 & a & 0 & e \\ 0 & 2b & -2c & 0 & f \\ h & i & j & k & g \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.3 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes K^8)$

$$IV\begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & (\alpha+\beta)f & f & d \\ 0 & 0 & -f & (\alpha+\beta)f & e \\ 0 & 0 & 0 & 0 & \beta f \end{bmatrix}.$$

4.0.11.4 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes Q^7)$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & d & e & 0 \\ 0 & 0 & 0 & d+f & e \\ 0 & 0 & 0 & 0 & d+2f \end{bmatrix}.$$

4.0.11.5 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes (\mathbb{R}^2 \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & \alpha d+\lambda f & e & d+\mu f \\ 0 & 0 & 0 & (\alpha+1)d+\lambda f & 0 \\ 0 & 0 & 0 & 0 & \alpha d+\mu f \end{bmatrix}.$$

4.0.11.6 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes (\mathbb{R}^2 \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & \alpha d+\lambda f & e & 0 \\ 0 & 0 & 0 & (\alpha+1)d+\lambda f & 0 \\ 0 & 0 & 0 & 0 & \beta d+\mu f \end{bmatrix}.$$

4.0.11.7 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes (\mathbb{R}^2 \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & f & 0 & d \\ 0 & 0 & 0 & f & e \\ 0 & 0 & 0 & 0 & f \end{bmatrix}.$$

4.0.11.8 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \oplus (\mathbb{R}^2 \rtimes \mathbb{R}^6))$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & f \end{bmatrix}.$$

4.0.11.9 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \oplus H^8)$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & (\beta+1)f & 0 & d \\ 0 & 0 & 0 & (\beta+1)f & e \\ 0 & 0 & 0 & 0 & \beta f \end{bmatrix}.$$

4.0.11.10 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \oplus H^8)$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & (\alpha+1)f & f & d \\ 0 & 0 & 0 & (\alpha+1)f & e \\ 0 & 0 & 0 & 0 & \alpha f \end{bmatrix}.$$

4.0.11.11 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \oplus H^8)$

$$IV \begin{bmatrix} a & b & g & i & k \\ c & -a & h & j & m \\ 0 & 0 & \alpha d+\beta e & d & f \\ 0 & 0 & 0 & \alpha d+\beta e & e \\ 0 & 0 & 0 & 0 & \alpha d+\beta e \end{bmatrix}.$$

4.0.11.12 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$IV \begin{bmatrix} a & b & 0 & i & k \\ c & -a & 0 & j & m \\ 0 & 0 & d+(\beta+1)e & f & h \\ 0 & 0 & 0 & d+\beta e & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}.$$

4.0.11.13 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^6)$

$$IV \begin{bmatrix} a & b & 0 & i & k \\ c & -a & 0 & j & m \\ 0 & 0 & f & d & e \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.11.14 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^6)$

$$IV \begin{bmatrix} a & b & 0 & i & k \\ c & -a & 0 & j & m \\ 0 & 0 & \alpha d + f & e & h \\ 0 & 0 & 0 & (1+\alpha)d + f & 0 \\ 0 & 0 & 0 & 0 & \alpha d + g \end{bmatrix}.$$

4.0.11.15 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^7)$

$$IV \begin{bmatrix} a & b & 0 & i & k \\ c & -a & 0 & j & m \\ 0 & 0 & (\alpha+1)e + \beta d & f & h \\ 0 & 0 & 0 & d + \alpha e + (\beta+1)d & g \\ 0 & 0 & 0 & 0 & d + \alpha e + \beta d \end{bmatrix}.$$

4.0.11.16 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^6 \rtimes A_{3.1})$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j + \beta k & j & m \\ 0 & 0 & 0 & \alpha j + \beta k & k \\ 0 & 0 & 0 & 0 & \alpha j + \beta k \end{bmatrix}.$$

4.0.11.17 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^6 \rtimes A_{3.2})$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & (\alpha+1)j & j & k \\ 0 & 0 & 0 & (\alpha+1)j & m \\ 0 & 0 & 0 & 0 & \alpha j \end{bmatrix}$$

4.0.11.18 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^6 \rtimes A_{3.3-3.5})$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & (\beta+1)j & 0 & k \\ 0 & 0 & 0 & (a+\beta)j & m \\ 0 & 0 & 0 & 0 & \beta j \end{bmatrix}.$$

4.0.11.19 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^6 \rtimes A_{3.6-3.7})$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & (a+\beta)j & j & k \\ 0 & 0 & -j & (a+\beta)j & m \\ 0 & 0 & 0 & 0 & \beta j \end{bmatrix}.$$

4.0.11.20 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (A_{3.3} \oplus A_{3.3} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j & 0 & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.21 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \rtimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & j & m \\ 0 & 0 & 0 & 0 & j \end{bmatrix}$$

4.0.11.22 $\lambda^2+\mu^2\neq 0\ \mathfrak{sl}(2,\mathbb{R})\ltimes(\mathbb{R}^2\ltimes(\mathbb{R}^2\oplus A_{5.1}))$

$$IV\begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j+\lambda m & k & 0 \\ 0 & 0 & 0 & (\alpha+1)j+\lambda m & 0 \\ 0 & 0 & 0 & 0 & \beta j+\mu m \end{bmatrix}$$

$$IV\begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j+\lambda m & k & 0 \\ 0 & 0 & 0 & (\alpha+1)j+\lambda m & 0 \\ 0 & 0 & 0 & 0 & \alpha j+\mu m \end{bmatrix}$$

$$IV\begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j+\lambda m & k & j+\mu m \\ 0 & 0 & 0 & (\alpha+1)j+\lambda m & 0 \\ 0 & 0 & 0 & 0 & \alpha j+\lambda m \end{bmatrix}$$

$$IV\begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j+\lambda m & k & 0 \\ 0 & 0 & 0 & (\alpha+1)j+\lambda m & 0 \\ 0 & 0 & 0 & 0 & (\alpha+1)j+\mu m \end{bmatrix}$$

$$IV\begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j+\lambda m & 0 & k \\ 0 & 0 & 0 & (\alpha+1)j+\lambda m & j+\mu m \\ 0 & 0 & 0 & 0 & (\alpha+1)j+\lambda m \end{bmatrix}$$

4.0.11.23 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \oplus N_7)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j & k & 0 \\ 0 & 0 & 0 & j+m & k \\ 0 & 0 & 0 & 0 & j+2m \end{bmatrix}$$

4.0.11.24 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^7)$

$$V \begin{bmatrix} d+a & b & f & g & h \\ c & d-a & i & -f & j \\ 0 & 0 & e+a & b & k \\ 0 & 0 & c & e-a & m \\ 0 & 0 & 0 & 0 & \lambda d + \mu e \end{bmatrix}$$

4.0.11.25 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes K^8)$

$$V \begin{bmatrix} d+a & b & e & f & g \\ c & d-a & h & i & j \\ 0 & 0 & d+a & b & k \\ 0 & 0 & c & d-a & m \\ 0 & 0 & 0 & 0 & \lambda d \end{bmatrix}$$

4.0.11.26 $\lambda^2+\mu^2\neq 0\,\mathfrak{sl}(2,\mathbb{R})\ltimes(\mathbb{R}^2\ltimes K^7)$

$$V\begin{bmatrix} \lambda d+a & b & f & g & h \\ c & \lambda d-a & i & -f & j \\ 0 & 0 & \mu d+a & b & k \\ 0 & 0 & c & \mu d-a & m \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.11.27 $\mathfrak{sl}(2,\mathbb{R})\ltimes(\mathbb{R}\ltimes K^8)$

$$V\begin{bmatrix} \lambda d+a & b & d+f & g & h \\ c & \lambda d-a & i & d-f & j \\ 0 & 0 & \lambda d+a & b & k \\ 0 & 0 & c & \lambda d-a & m \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.11.28 $\mathfrak{sl}(2,\mathbb{R})\ltimes(\mathbb{R}\ltimes K^8)$

$$V\begin{bmatrix} (\lambda+1)d+a & b & e & f & g \\ c & (\lambda+1)d-a & h & i & j \\ 0 & 0 & \lambda d+a & b & k \\ 0 & 0 & c & \lambda d-a & m \\ 0 & 0 & 0 & 0 & \mu d \end{bmatrix}$$

4.0.11.29 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes (A_{3.3} \oplus A_{5.7a=b=c=1}))$

$$V \begin{bmatrix} a+d & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & 0 \\ 0 & 0 & c & e-a & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.30 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes K^7)$

$$V \begin{bmatrix} \lambda d+a & b & e & f & g \\ c & \lambda d-a & h & -e & i \\ 0 & 0 & a+\mu d & b & j \\ 0 & 0 & c & \mu d-a & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.31 $\mathfrak{so}(3) \rtimes (\mathbb{R}^2 \rtimes N^7)$

$$VII \begin{bmatrix} 0 & c & -b & d & h \\ -c & 0 & a & e & i \\ b & -a & 0 & f & j \\ 0 & 0 & 0 & g & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

4.0.11.32 $\mathfrak{so}(3) \rtimes (\mathbb{R}^2 \rtimes H^7)$

$$VII \begin{bmatrix} 0 & c & -b & 0 & d \\ -c & 0 & a & 0 & e \\ b & -a & 0 & 0 & f \\ h & i & j & k & g \\ 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

4.0.11.33 $\mathbb{R} \oplus (\mathfrak{sl}(3,\mathbb{R}) \rtimes \mathbb{R}^3)$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.11.34 $\mathfrak{sl}(3,\mathbb{R}) \rtimes A_{4.5a=b=1}$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ 0 & 0 & 0 & m & m \\ 0 & 0 & 0 & 0 & m \end{bmatrix} X \begin{bmatrix} a & b & c & i & 0 \\ d & e & f & j & 0 \\ g & h & -(a+e) & k & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & \lambda m \end{bmatrix}$$

4.0.11.35 $\mathbb{R} \oplus (\mathfrak{so}(3,1) \rtimes \mathbb{R}^4)$

$$XII \begin{bmatrix} g & a & b & c & h \\ -a & g & d & e & i \\ -b & -d & g & f & j \\ c & e & f & g & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

4.0.11.36 $\mathbb{R} \oplus \mathfrak{sp}(4)$

$$XIII \begin{bmatrix} a+k & b & e & f & 0 \\ c & d+k & f & g & 0 \\ h & i & k-a & -c & 0 \\ i & j & -b & k-d & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

4.0.11.37 $\lambda^2 + \mu^2 \neq 0 \, \mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes \mathbb{R}^4)$

$$XVIII \begin{bmatrix} a+\lambda k & b & g & h & 0 \\ c & \lambda k - a & i & j & 0 \\ 0 & 0 & d+\mu k & e & 0 \\ 0 & 0 & f & \mu k - d & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.38 $\mathbb{R}^2 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes \mathbb{R}^4)$

$$XVIII \begin{bmatrix} a+k & b & 0 & 0 & g \\ c & k-a & 0 & 0 & h \\ 0 & 0 & d+k & e & i \\ 0 & 0 & f & k-d & j \\ 0 & 0 & 0 & 0 & \lambda k + \mu m \end{bmatrix}$$

4.0.11.39 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes \mathbb{R}^6$

$$XVIII \begin{bmatrix} a & b & g & h & k \\ c & -a & i & j & m \\ 0 & 0 & d & e & 0 \\ 0 & 0 & f & -d & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.11.40 $\mathbb{R} \oplus ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes A_{5.7a=b=c=1})$

$$XIX \begin{bmatrix} a+d+g & b & e & 0 & h \\ c & -a+d+g & 0 & e & i \\ f & 0 & a-d+g & b & j \\ 0 & f & c & g-a-d & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.11.41 $\lambda^2 + \mu^2 \neq 0 \, \mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}))$

$$XXI \begin{bmatrix} a+\lambda m & b & 0 & 0 & 0 \\ c & +\lambda m - a & 0 & 0 & 0 \\ 0 & 0 & \mu m + d & e & f \\ 0 & 0 & g & \mu m + h & i \\ 0 & 0 & j & k & \mu m - (d+h) \end{bmatrix}$$

4.0.11.42 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \rtimes \mathbb{R}^6$

$$XXII \begin{bmatrix} a & b & g & h & i \\ c & -a & j & k & m \\ 0 & 0 & 0 & d & -e \\ 0 & 0 & -d & 0 & f \\ 0 & 0 & e & -f & 0 \end{bmatrix}$$

4.0.11.43 $\mathbb{R} \oplus (\mathfrak{so}(4) \rtimes A_{5.7a=b=c=1})$

$$XXIII \begin{bmatrix} g & a & b & c & h \\ -a & g & d & e & i \\ -b & -d & g & f & j \\ -c & -e & -f & g & k \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.12 Dimension 13

4.0.12.1 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes K^8)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & \lambda g + \mu d & d & e \\ 0 & 0 & -d & \lambda g + \mu d & f \\ 0 & 0 & 0 & 0 & (\lambda+1)g + \mu d \end{bmatrix}.$$

4.0.12.2 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes K^8)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & d + (\lambda+\mu)g & g & e \\ 0 & 0 & -g & d + (\lambda+\mu)g & f \\ 0 & 0 & 0 & 0 & d + \mu g \end{bmatrix}.$$

4.0.12.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^9)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & (\lambda+\mu+1)d & e & g \\ 0 & 0 & 0 & (\lambda+\mu)d & f \\ 0 & 0 & 0 & 0 & \lambda d \end{bmatrix}.$$

4.0.12.4 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes N^8)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & d+(\lambda+1)g & g & e \\ 0 & 0 & 0 & d+(\lambda+1)g & f \\ 0 & 0 & 0 & 0 & d+\lambda g \end{bmatrix}.$$

4.0.12.5 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes N^9)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & d+\lambda e+\mu f & e & g \\ 0 & 0 & 0 & d+\lambda e+\mu f & f \\ 0 & 0 & 0 & 0 & d+\lambda e+\mu f \end{bmatrix}.$$

4.0.12.6 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes N^8)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & \lambda e+\mu g & f & d \\ 0 & 0 & 0 & (\lambda+1)e+\mu g & 0 \\ 0 & 0 & 0 & 0 & \lambda e+\nu g \end{bmatrix}.$$

4.0.12.7 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \rtimes A_{5.1} \oplus A_{3.3})$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix}.$$

4.0.12.8 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \rtimes N^8)$

$$IV \begin{bmatrix} a & b & h & j & m \\ c & -a & i & k & n \\ 0 & 0 & \lambda e + g & f & d \\ 0 & 0 & 0 & (\lambda+1)e & 0 \\ 0 & 0 & 0 & 0 & \lambda e + g \end{bmatrix}.$$

4.0.12.9 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^3 \rtimes N^7)$

$$IV \begin{bmatrix} a & b & 0 & j & m \\ c & -a & 0 & k & n \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & d \\ 0 & 0 & 0 & 0 & e \end{bmatrix}.$$

4.0.12.10 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^8)$

$$IV \begin{bmatrix} a & b & h & i & j \\ c & -a & k & m & n \\ 0 & 0 & d+(\mu+1)g & 0 & e \\ 0 & 0 & 0 & d+(\lambda+\mu)g & f \\ 0 & 0 & 0 & 0 & d+\mu f \end{bmatrix}.$$

4.0.12.11 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes N^9)$

$$IV \begin{bmatrix} a & j & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & (\alpha+b+1)j & k & m \\ 0 & 0 & 0 & (\alpha+b)j & n \\ 0 & 0 & 0 & 0 & \alpha j \end{bmatrix}$$

4.0.12.12 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \lambda j+\mu k & k & m \\ 0 & 0 & -w & \lambda j+\mu k & n \\ 0 & 0 & 0 & 0 & (\lambda+1)j+\mu k \end{bmatrix}.$$

4.0.12.13 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes (\mathbb{R}^2 \oplus A_{5.1}))$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & n \\ 0 & 0 & j & w & 0 \\ 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}.$$

4.0.12.14 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j + k & m & n \\ 0 & 0 & 0 & (\alpha+1)j + k & 0 \\ 0 & 0 & 0 & 0 & \alpha j + k \end{bmatrix}.$$

4.0.12.15 $\lambda^2 + \mu^2 \neq 0 \, \mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & \alpha j + \lambda k & m & n \\ 0 & 0 & 0 & (\alpha+1)j + \lambda k & 0 \\ 0 & 0 & 0 & 0 & \alpha j + \mu k \end{bmatrix}.$$

4.0.12.16 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \ltimes N^9)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j+\alpha k+\beta m & k & n \\ 0 & 0 & 0 & j+\alpha k+\beta m & m \\ 0 & 0 & 0 & 0 & j+\alpha k+\beta m \end{bmatrix}.$$

4.0.12.17 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j+(\lambda+1)k & k & m \\ 0 & 0 & 0 & j+(\lambda+1)k & n \\ 0 & 0 & 0 & 0 & j+\lambda k \end{bmatrix}.$$

4.0.12.18 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j+(\beta+1)k & 0 & m \\ 0 & 0 & 0 & j+(\lambda+\beta)k & n \\ 0 & 0 & 0 & 0 & j+\beta k \end{bmatrix}.$$

4.0.12.19 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j+(\lambda+\beta)k & k & m \\ 0 & 0 & -k & j+(\lambda+\beta)k & n \\ 0 & 0 & 0 & 0 & w+\beta k \end{bmatrix}$$

4.0.12.20 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R} \rtimes K^8))$

$$V \begin{bmatrix} a & b & f & g & h \\ c & d & i & j & k \\ 0 & 0 & a & b & m \\ 0 & 0 & c & d & n \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.12.21 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes K^8)$

$$V \begin{bmatrix} (\lambda+1)d+a & b & f & g & h \\ c & (\lambda+1)d-a & i & j & k \\ 0 & 0 & \lambda d+a & b & m \\ 0 & 0 & c & \lambda d-a & n \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$V \begin{bmatrix} a+d & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & m \\ 0 & 0 & c & e-a & n \\ 0 & 0 & 0 & 0 & \lambda d+\mu e \end{bmatrix}$$

4.0.12.22 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^2 \ltimes K^8)$

$$V\begin{bmatrix} a+\lambda d & b & f & g & h \\ c & \lambda d - a & i & j & k \\ 0 & 0 & a+\mu d & b & m \\ 0 & 0 & c & \mu d - a & n \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$$

4.0.12.23 $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes K^7)$

$$V\begin{bmatrix} a+d & b & f & g & h \\ c & d-a & i & -f & j \\ 0 & 0 & a+e & b & k \\ 0 & 0 & c & e-a & m \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.24 $\mathfrak{sl}(3,\mathbb{R}) \ltimes A_{5.7a=b=c=-\lambda}$

$$X\begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ 0 & 0 & 0 & (\lambda+1)n & m \\ 0 & 0 & 0 & 0 & \lambda n \end{bmatrix}$$

4.0.12.25 $\mathbb{R} \oplus (\mathfrak{sl}(3,\mathbb{R}) \rtimes A_{4.5a=b=1})$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ 0 & 0 & 0 & n & m \\ 0 & 0 & 0 & 0 & n \end{bmatrix} X \begin{bmatrix} a & b & c & i & 0 \\ d & e & f & j & 0 \\ g & h & -(a+e) & k & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.26 $\mathbb{R}^3 \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^4)$

$$XVIII \begin{bmatrix} a & b & e & f & 0 \\ c & d & g & h & 0 \\ 0 & 0 & i & j & 0 \\ 0 & 0 & k & m & 0 \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.27 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2) \oplus (\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^2)$

$$XVIII \begin{bmatrix} a & b & 0 & 0 & e \\ c & d & 0 & 0 & f \\ 0 & 0 & i & j & g \\ 0 & 0 & k & m & h \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.28 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^6)$

$$XVIII \begin{bmatrix} a+n & b & g & h & k \\ c & n-a & i & j & m \\ 0 & 0 & d+n & e & 0 \\ 0 & 0 & f & n-d & 0 \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.29 $((\lambda-\mu)^2 + (\lambda-\nu)^2 + (\mu-\nu)^2 \neq 0)\, (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes (\mathbb{R} \ltimes \mathbb{R}^6)$

$$XVIII \begin{bmatrix} a+\lambda n & b & g & h & k \\ c & -a+\lambda n & i & j & m \\ 0 & 0 & d+\mu n & e & 0 \\ 0 & 0 & f & -d+\mu n & 0 \\ 0 & 0 & 0 & 0 & \nu n \end{bmatrix}$$

4.0.12.30 $\mathbb{R} \oplus ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes \mathbb{R}^6)$

$$XVIII \begin{bmatrix} n+a & b & g & h & k \\ c & n-a & i & j & m \\ 0 & 0 & n+d & e & 0 \\ 0 & 0 & f & n-d & 0 \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.12.31 $((\lambda-\mu)^2+(\lambda-\nu)^2+(\mu-\nu)^2\neq 0)\,(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\rtimes(\mathbb{R}\rtimes\mathbb{R}^6)$

$$XVIII\begin{bmatrix}\lambda n+a & b & g & h & k\\ c & \lambda n-a & i & j & m\\ 0 & 0 & \mu n+d & e & 0\\ 0 & 0 & f & \mu n-d & 0\\ 0 & 0 & 0 & 0 & \nu n\end{bmatrix}$$

4.0.12.32 $((\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\rtimes(\mathbb{R}\rtimes\mathbb{R}^6)$

$$XX\begin{bmatrix}a+\lambda n & b & g & h & i\\ c & +\lambda n-a+y & j & k & m\\ 0 & 0 & 2d+\mu n & 2e & 0\\ 0 & 0 & f & \mu n & e\\ 0 & 0 & 0 & 2f & \mu n-2d\end{bmatrix}$$

4.0.12.33 $\mathbb{R}^2\oplus(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(3,\mathbb{R}))$

$$XXI\begin{bmatrix}a & b & 0 & 0 & 0\\ c & d & 0 & 0 & 0\\ 0 & 0 & e & f & g\\ 0 & 0 & h & i & j\\ 0 & 0 & k & m & n\end{bmatrix}$$

4.0.12.34 $\lambda^2 + \mu^2 \neq 0 \, ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$XXII \begin{bmatrix} a+\lambda n & b & g & h & i \\ c & \lambda n - a & j & k & m \\ 0 & 0 & \mu n & d & -e \\ 0 & 0 & -d & \mu n & f \\ 0 & 0 & e & -f & \mu n \end{bmatrix}$$

4.0.13 Dimension 14

4.0.13.1 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes K^8)$

$$IV \begin{bmatrix} a & b & i & k & n \\ c & -a & j & m & p \\ 0 & 0 & h & e & f \\ 0 & 0 & -e & h & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}.$$

4.0.13.2 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^9)$

$$IV \begin{bmatrix} a & b & i & k & n \\ c & -a & j & m & p \\ 0 & 0 & d+(\beta+1)e & f & h \\ 0 & 0 & 0 & d+\beta e & g \\ 0 & 0 & 0 & 0 & d \end{bmatrix}.$$

4.0.13.3 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes N^8)$

$$IV \begin{bmatrix} a & b & i & k & n \\ c & -a & j & m & p \\ 0 & 0 & f & d & e \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix}.$$

4.0.13.4 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^9)$

$$IV \begin{bmatrix} a & b & i & k & n \\ c & -a & j & m & p \\ 0 & 0 & (\lambda+1)e+\mu d & f & h \\ 0 & 0 & 0 & \lambda e+(\mu+1)d & g \\ 0 & 0 & 0 & 0 & \lambda e+\mu d \end{bmatrix}.$$

4.0.13.5 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes N^9)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j+(\beta+1)k & m & n \\ 0 & 0 & 0 & j+\beta k & p \\ 0 & 0 & 0 & 0 & j \end{bmatrix}$$

4.0.13.6 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes K^8)$

$$IV \begin{bmatrix} a & b & d & e & f \\ c & -a & g & h & i \\ 0 & 0 & j & k & m \\ 0 & 0 & -k & j & n \\ 0 & 0 & 0 & 0 & p \end{bmatrix}$$

4.0.13.7 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes K^8)$

$$V \begin{bmatrix} a+d & b & f & g & h \\ c & d-a & i & j & k \\ 0 & 0 & a+e & b & m \\ 0 & 0 & c & e-a & n \\ 0 & 0 & 0 & 0 & p \end{bmatrix}$$

4.0.13.8 $(\mathfrak{sl}(3,\mathbb{R}) \rtimes (A_{2.1} \oplus A_{4.5a=b=1})$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ 0 & 0 & 0 & m & p \\ 0 & 0 & 0 & 0 & n \end{bmatrix}$$

4.0.13.9 $\mathfrak{sl}(3,\mathbb{R}) \rtimes \mathbb{R}^6$

$$X \begin{bmatrix} a & b & c & i & m \\ d & e & f & j & n \\ g & h & -(a+e) & k & p \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.13.10 $\mathfrak{sp}(4) \rtimes \mathbb{R}^4$

$$XIII \begin{bmatrix} a & b & e & f & k \\ c & d & f & g & m \\ h & i & -a & -c & n \\ i & j & -b & -d & p \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.0.13.11 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$XVIII \begin{bmatrix} a+k & b & c & d & e \\ f & k-a & g & h & i \\ 0 & 0 & p+j & m & 0 \\ 0 & 0 & n & p-j & 0 \\ 0 & 0 & 0 & 0 & \lambda k + \mu p \end{bmatrix}$$

4.0.13.12 $\lambda^2+\mu^2\neq 0\,(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\ltimes(\mathbb{R}^2\rtimes\mathbb{R}^6)$

$$XVIII\begin{bmatrix} a+\lambda n & b & c & d & e \\ f & \lambda n-a & g & h & i \\ 0 & 0 & j+\mu n & k & 0 \\ 0 & 0 & m & \mu n-j & 0 \\ 0 & 0 & 0 & 0 & p \end{bmatrix}$$

4.0.13.13 $(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\ltimes K_8$

$$XVIII\begin{bmatrix} a & b & g & h & k \\ c & -a & i & j & m \\ 0 & 0 & d & e & n \\ 0 & 0 & f & -d & p \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.13.14 $(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\ltimes(\mathbb{R}^2\rtimes\mathbb{R}^6)$

$$XX\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & 2k+p & 2m & 0 \\ 0 & 0 & n & p & m \\ 0 & 0 & 0 & 2n & p-2k \end{bmatrix}$$

4.0.13.15 $((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$XXII \begin{bmatrix} a+n & b & g & h & i \\ c & n-a & j & k & m \\ 0 & 0 & p & d & -e \\ 0 & 0 & -d & p & f \\ 0 & 0 & e & -f & p \end{bmatrix}$$

4.0.14 Dimension 15

4.0.14.1 $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes N_9)$

$$IV \begin{bmatrix} a & b & j & m & p \\ c & -a & k & n & q \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & d \\ 0 & 0 & 0 & 0 & e \end{bmatrix}.$$

4.0.14.2 $(\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$X \begin{bmatrix} a & b & c & i & m \\ d & e & f & j & n \\ g & h & -(a+e) & k & p \\ 0 & 0 & 0 & \lambda q & q \\ 0 & 0 & 0 & -q & \lambda q \end{bmatrix}$$

4.0.14.3 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$X \begin{bmatrix} a & b & c & i & m \\ d & e & f & j & n \\ g & h & -(a+e) & k & p \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & \lambda q \end{bmatrix}$$

4.0.14.4 $\mathfrak{sl}(3,\mathbb{R}) \rtimes N^7$

$$X \begin{bmatrix} a & b & c & i & m \\ d & e & f & j & n \\ g & h & -(a+e) & k & p \\ 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.14.5 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$X \begin{bmatrix} a & b & c & i & m \\ d & e & f & j & n \\ g & h & -(a+e) & k & p \\ 0 & 0 & 0 & q & q \\ 0 & 0 & 0 & 0 & q \end{bmatrix}$$

4.0.14.6 $\mathfrak{sl}(3,\mathbb{R}) \rtimes H^7$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ m & n & p & 0 & q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.14.7 $\mathbb{R} \oplus (\mathfrak{sp}(4) \rtimes \mathbb{R}^4)$

$$XIII \begin{bmatrix} a+k & b & e & f & m \\ c & d+k & f & g & n \\ h & i & k-a & -c & p \\ i & j & -b & k-d & q \\ 0 & 0 & 0 & 0 & k \end{bmatrix}.$$

4.0.14.8 $(\lambda \neq \mu)\, \mathfrak{sp}(4) \rtimes A_{5.7a=b=c=\lambda-\mu}$

$$XIII \begin{bmatrix} a+\lambda k & b & e & f & m \\ c & d+\lambda k & f & g & n \\ h & i & \lambda k-a & -c & p \\ i & j & -b & \lambda k-d & q \\ 0 & 0 & 0 & 0 & \mu k \end{bmatrix}.$$

4.0.14.9 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes (\mathbb{R}^3 \rtimes \mathbb{R}^6)$

$$XVIII \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & k & m & 0 \\ 0 & 0 & n & p & 0 \\ 0 & 0 & 0 & 0 & q \end{bmatrix}$$

4.0.14.10 $\lambda^2 + \mu^2 + \nu^2 \neq 0\, (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \ltimes (\mathbb{R} \rtimes K^8)$

$$XVIII \begin{bmatrix} a+\lambda q & b & g & h & k \\ c & \lambda q - a & i & j & m \\ 0 & 0 & d+\mu q & e & n \\ 0 & 0 & f & \mu q - d & p \\ 0 & 0 & 0 & 0 & \nu q \end{bmatrix}$$

4.0.15 Dimension 16

4.0.15.1 $(\mathfrak{sl}(3,\mathbb{R}) \ltimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$X \begin{bmatrix} a & b & c & i & n \\ d & e & f & j & p \\ g & h & -(a+e) & k & q \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & r \end{bmatrix}$$

4.0.15.2 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$X \begin{bmatrix} a & b & c & i & n \\ d & e & f & j & p \\ g & h & -(a+e) & k & q \\ 0 & 0 & 0 & m & r \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

4.0.15.3 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes \mathbb{R}^6)$

$$X \begin{bmatrix} a & b & c & k & m \\ d & e & f & n & p \\ g & h & -(a+e) & q & r \\ 0 & 0 & 0 & i & j \\ 0 & 0 & 0 & -j & i \end{bmatrix}$$

4.0.15.4 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes N^7)$

$$X \begin{bmatrix} a & b & c & i & n \\ d & e & f & j & p \\ g & h & -(a+e) & k & q \\ 0 & 0 & 0 & (\lambda+1)m & r \\ 0 & 0 & 0 & 0 & \lambda m \end{bmatrix}$$

4.0.15.5 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes H^7)$

$$X\begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ m & n & p & \lambda r & q \\ 0 & 0 & 0 & 0 & \mu r \end{bmatrix}$$

4.0.15.6 $\lambda^2 + \mu^2 \neq 0\, \mathbb{R} \oplus \mathfrak{sl}(4,\mathbb{R})$

$$XI\begin{bmatrix} a+\lambda r & b & c & d & 0 \\ e & f+\lambda r & g & h & 0 \\ i & j & k+\lambda r & m & 0 \\ n & p & q & -(a+f+k)+\lambda r & 0 \\ 0 & 0 & 0 & 0 & \mu r \end{bmatrix}$$

4.0.15.7 $\mathfrak{sp}(4) \rtimes (\mathbb{R} \oplus A_{5.7a=b=c=1})$

$$XIII\begin{bmatrix} a+k & b & e & f & m \\ c & d+k & f & g & n \\ h & i & k-a & -c & p \\ i & j & -b & k-d & q \\ 0 & 0 & 0 & 0 & r \end{bmatrix}.$$

4.0.15.8 $\lambda^2+\mu^2\neq 0\,(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\ltimes(\mathbb{R}^2\ltimes K^8)$

$$XVIII\begin{bmatrix}\lambda q+a & b & g & h & i\\ c & \lambda q-a & j & k & m\\ 0 & 0 & \mu q+d & e & n\\ 0 & 0 & f & \mu q-d & p\\ 0 & 0 & 0 & 0 & r\end{bmatrix}$$

4.0.15.9 $(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\ltimes(\mathbb{R}^2\ltimes K^8)$

$$XVIII\begin{bmatrix}a+n & b & c & d & e\\ f & n-a & g & h & i\\ 0 & 0 & j+p & k & q\\ 0 & 0 & m & p-j & r\\ 0 & 0 & 0 & 0 & \lambda n+\mu p\end{bmatrix}$$

4.0.16 Dimension 17

4.0.16.1 $\mathfrak{sl}(3,\mathbb{R})\ltimes(\mathbb{R}^2\ltimes N^7)$

$$X\begin{bmatrix}a & b & c & i & n\\ d & e & f & j & p\\ g & h & -(a+e) & k & q\\ 0 & 0 & 0 & m & r\\ 0 & 0 & 0 & 0 & s\end{bmatrix}$$

4.0.16.2 $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R}^2 \rtimes H^7)$

$$X \begin{bmatrix} a & b & c & 0 & i \\ d & e & f & 0 & j \\ g & h & -(a+e) & 0 & k \\ m & n & p & q & r \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

4.0.16.3 $\mathbb{R}^2 \oplus \mathfrak{sl}(4,\mathbb{R})$

$$XI \begin{bmatrix} a & b & c & d & 0 \\ e & f & g & h & 0 \\ i & j & k & m & 0 \\ n & p & q & r & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

4.0.16.4 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \rtimes (\mathbb{R}^3 \rtimes K^8)$

$$XVIII \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & k & m & n \\ 0 & 0 & p & q & r \\ 0 & 0 & 0 & 0 & s \end{bmatrix}$$

4.0.16.5 $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})) \rtimes \mathbb{R}^6$

$$XXI \begin{bmatrix} a & b & m & n & p \\ c & -a & q & r & s \\ 0 & 0 & d & e & f \\ 0 & 0 & g & h & i \\ 0 & 0 & j & k & -(d+h) \end{bmatrix}$$

4.0.17 Dimension 18

4.0.17.1 $\lambda^2 + \mu^2 \neq 0\, (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6)$

$$XXI \begin{bmatrix} a+\lambda t & b & m & n & p \\ c & \lambda t - a & q & r & s \\ 0 & 0 & \mu t + d & e & f \\ 0 & 0 & g & \mu t + h & i \\ 0 & 0 & j & k & \mu t - (d+h) \end{bmatrix}$$

4.0.18 Dimension 19

4.0.18.1 $\mathfrak{sl}(4,\mathbb{R}) \rtimes \mathbb{R}^4$

$$XI \begin{bmatrix} a & b & c & d & r \\ e & f & g & h & s \\ i & j & k & m & t \\ n & p & q & -(a+f+k) & u \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.0.18.2 $\mathbb{R} \oplus ((\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R} \rtimes \mathbb{R}^6))$

$$XXI \begin{bmatrix} a & b & p & q & r \\ c & d & s & t & u \\ 0 & 0 & e & f & g \\ 0 & 0 & h & i & j \\ 0 & 0 & k & m & n \end{bmatrix}$$

4.0.18.3 $\mathbb{R} \oplus (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)) \rtimes \mathbb{R}^6)$

$$XXII \begin{bmatrix} g & b & p & q & r \\ c & g-a & s & t & u \\ 0 & 0 & g & d & -e \\ 0 & 0 & -d & g & f \\ 0 & 0 & e & -f & g \end{bmatrix}$$

4.0.19 Dimension 20

4.0.19.1 $(\lambda \neq \mu)\, \mathfrak{sl}(4,\mathbb{R}) \rtimes A_{5.7a=b=c=1}$

$$XI \begin{bmatrix} \lambda a + r & b & c & d & s \\ e & f + \lambda r & g & h & t \\ i & j & k + \lambda r & m & u \\ n & p & q & \lambda r - (a + f + k) & v \\ 0 & 0 & 0 & 0 & \mu r \end{bmatrix}$$

4.0.19.2 $\mathbb{R} \oplus (\mathfrak{sl}(4,\mathbb{R}) \rtimes \mathbb{R}^4)$

$$XI \begin{bmatrix} a+r & b & c & d & s \\ e & f+r & g & h & t \\ i & j & k+r & m & u \\ n & p & q & r-(a+f+k) & v \\ 0 & 0 & 0 & 0 & r \end{bmatrix}$$

4.0.20 Dimension 21

4.0.20.1 $\mathbb{R} \oplus (\mathfrak{sl}(4,\mathbb{R}) \rtimes A_{5.7a=b=c=1})$

$$XI \begin{bmatrix} a & b & c & d & s \\ e & f & g & h & t \\ i & j & k & m & u \\ n & p & q & r & v \\ 0 & 0 & 0 & 0 & w \end{bmatrix}$$

4.0.21 Dimension 25

4.0.21.1 $\mathfrak{sl}(5,\mathbb{R}) \oplus \mathbb{R} \approx \mathfrak{gl}(5,\mathbb{R})$

$$XIV \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & m & n & p & q \\ r & s & t & u & v \\ w & x & y & z & \theta \end{bmatrix}$$

4.1 Appendix

Here we give several nilpotent matrix subalgebras of $\mathfrak{gl}(5,\mathbb{R})$ that occur in several Levi subalgebras of $\mathfrak{gl}(5,\mathbb{R})$.

4.1.0.2 H^7

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ d & e & f & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.3 K^7

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & -a & e \\ 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.4 N^7

$$\begin{bmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.5 Q^7

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.6 K^8

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.7 N^8

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.8 P^8

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & -a & e \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.1.0.9 N^9

$$\begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

References

[Dyn] E. B. Dynkin, *The maximal subgroups of the classical groups*, Amer. Math. Soc.: Translation Series 2(**6**), (1957), 245-378.

[GTT] R. Ghanam, G. Thompson and S. Tonon, *Representations for six-dimensional nilpotent Lie algebras*, Hadronic J. 29 (2006), **29**(3), 299–317.

[GT] R. Ghanam and G. Thompson, *Minimal matrix representations of five-dimensional Lie algebras*, Extracta Mathematicae, **30**(1), (2015), 95-133.

[Hel] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press (1978).

[Hum] J. Humphreys, *Lie Algebras and their Representations*, Springer (1997).

[L] S. Lie (1893), *Theorie der Transformationsgruppen*, Leipzig, §§136-137, pp. 713-732.

[Mor] V. V. Morozov, *Classification of Nilpotent Lie Algebras in dimension six*, Izv. Vysshikh Uchebn. **(4)**(5) (1958),161-171.

[Mub1] G. M. Mubarakzyanov, *On solvable Lie algebras*, Izv. Vysshikh Uchebn. Zavedenii Mat. **1**(32) (1963), 114–123.

[Mub2] G. M. Mubarakzyanov, *Classification of real Lie Algebras in dimension five*, Izv. Vysshikh Uchebn. Zavedenii Mat. **3**(34) (1963), 99–106.

[Mub3] G. M. Mubarakzyanov, *Classification of Solvable Lie Algebras in dimension six with one non-nilpotent basis element*, Izv. Vysshikh Uchebn. Zavedenii Mat. **4**(35) (1963), 104–116.

[Jac] N. Jacobson, *Schur's Theorems on Commutative Matrices*, Bulletin of the American Mathematical Society **50**, 431-6,(1944).

[PSWZ] J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus, *Invariants of real low dimension Lie algebras* , J. Math Phys **17** (1976), 986-994.

[ST] A. Shabanskaya and G. Thompson, *Six-dimensional Lie algebras with a five-dimensional nilradical*, J. of Lie Theory, (2013),**23**(3), 313-355.

[SW] L. Snobl and P. Winternitz, *Classification and Identification of Lie Algebras*, American Mathematical Society ,(2015), Providence, R.I.

[TW] G. Thompson and Z. Wick, *Subalgebras of* $\mathfrak{gl}(3,\mathbb{R})$, Extracta Mathematica, **27**(2), 201-230, (2012).

[Tur1] P Turkowski, *Low-dimensional real Lie Algebras*, J. Math. Phys. **29 (10)** (1988), 2139-2144.

[Tur2] P Turkowski, *Solvable Lie Algebras of dimension six*, J. Math. Phys. **31 (6)** (1990), 1344-1350.

[Um] K.A. Umlauf, *Uber die Zusammensetzung der endlichen continuierliche Transformationgruppen insbesondere der Gruppen von Rang null*, doctoral dissertation, University of Leipzig, 1891.