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An isospectral flow for complex upper Hessenberg matrices

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An Isospectral Flow for Complex Upper Hessenberg Matrices

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics

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An Abstract of

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We study an isospectral flow (Lax flow) that provides an explicit deformation from upper Hessenberg complex matrices to normal matrices, extending to the complex case and to the case of normal matrices the results of [1]. The Lax flow is given by

$$\frac{dA}{dt} = [[A^{\dagger}, A]_{du}, A],$$

where brackets indicate the usual matrix commutator, $[A, B] := AB - BA$, $A^{\dagger}$ is the conjugate transpose of A and the matrix $[A^{\dagger}, A]_{du}$ is the matrix equal to $[A^{\dagger}, A]$ along diagonal and upper triangular entries and zero below diagonal. We prove that if the initial condition A_0 is an upper Hessenberg matrix with simple spectrum, then $\lim_{t\to+\infty} A(t)$ exists and it is a normal upper Hessenberg matrix isospectral to A_0 and if the spectrum of A_0 is contained in a line in the complex plane, then the ω-limit set is actually a tridiagonal normal matrix.

Furthermore, we show that this flow is also the solution of an infinite time horizon optimal control problem and we prove that it can be used to construct even dimensional real skew-symmetric tridiagonal matrices with given simple spectrum, and with given signs pattern for the codiagonal elements.

I dedicate this dissertation to my son, daughter, wife and my parents.]

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Contents

List of Abbreviations

det .... Determinant

HJB .... Hamilton-Jacobi-Bellman

Im .... Imaginary part

lpdf .... Locally positive definite function

ODE .... Ordinary Differential Equation

PDE .... Partial Differential Equation
pdf .... Positive definite function

Re .... Real part

Spect .... Spectrum
SVD .... Singular value decomposition

List of Symbols

$\mathbb{R}$ .... Set of real numbers
$\mathbb{C}$ .... Set of complex numbers
$\mathbb{R}^{m\times n}$ .... Space of real matrices of order $m \times n$
$\mathbb{C}^{m\times n}$ .... Space of complex matrices of order $m \times n$
$M_n(\mathbb{R})$ ...... Space of real matrices of order $n \times n$
$M_n(\mathbb{C})$ ...... Space of complex matrices of order $n \times n$
$\mathbb{O}$ .... Zero matrix of order $n \times n$
$\mathbb{I}$ .... Identity matrix of order $n \times n$
$A(t)$ .... Matrix of order $n \times n$ in time t
A_0 .... $A(0)$
A_d .... The matrix equal to A along diagonal entries and zero everywhere else
A_u .... The matrix equal to A along strictly upper diagonal elements and zero everywhere else
A_l .... The matrix equal to A along strictly lower diagonal elements and zero everywhere else
A_{du} .... The matrix equal to A along diagonal and strictly upper diagonal elements and zero everywhere else
A_{dl} .... The matrix equal to A along diagonal and strictly lower diagonal elements and zero everywhere else
A^T .... The transposition of A
$\bar{A}$ .... The complex conjugation of A
$A^\dagger$ .... The conjugate transposition of A
A^{-1} .... The inverse of A
$\dot{A}$ .... $\frac{dA}{dt}$
$[A, B]$ .... Lie bracket of A and B or $AB - BA$
H^+ .... The vector space of upper Hessenberg matrices
$\mathcal{G}_0$ .... The connected component containing the identity of the Lie group of invertible upper triangular matrices with determinant equal to one
$\mathfrak{g}_0$ .... The corresponding Lie algebra, consisting of upper triangular matrices with trace equal to zero
$\Omega(A)$ .... Omega limit set of A
Λ .... Spectrum of A_0
$\mathcal{N}_\Lambda$ .... The manifold consisting of all normal matrices isospectral to A_0
$\mathcal{T}_\Lambda$ .... Space of normal tridiagonal matrices isospectral to A_0

Chapter 1

Introduction

In this chapter we are going to discuss about the flow under study, and present the result we proved so far. Isospectral flows (Lax equations) appear as ODEs of the following form:

$$\frac{dA}{dt} = [B, A], \tag{1.0.1}$$

where A is a $n \times n$ matrix, B is a matrix function whose entries are functions of the entries of A, $[A, B] := AB - BA$ is the matrix commutator. The pair (B, A) is usually called a Lax pair. If the matrix function B has entries that are C^1 functions of the entries of A, the standard existence and uniqueness theorem for ODEs shows that the Cauchy problem for (1.0.1) has locally a solution and this solution is unique. In our case, the entries of B will be quadratic polynomials in the entries of A.

Isospectral flows have a relatively long history and they appear in a variety of fields, from integrable systems (see [20], [33], [7], [9]), to representation theory via coadjoint orbits (see [3]). They also appear in numerical linear algebra, once it was realized that suitably constructed isospectral flows provide a continuous interpolation for discrete algorithms like the QR-factorization (see [34]). This area of research has been vastly expanded in subsequent years, including also the realization of continuous algorithms (flows of ODEs) for which no discrete version is currently available.

In this thesis we extend the results of [1] to the case of complex upper Hessenberg matrices and to the case of real upper Hessenberg matrices with complex spectrum. Let us underline that the proof of convergence is substantially different from the case of [1], essentially because the normality condition for a matrix is a nonlinear condition, while the property of being symmetric, i.e., the case analyzed in [1] is a linear condition on the entries of a matrix. This makes the proof of convergence and the estimates involved more delicate. Besides, [1] was dealing only with real matrices with real spectrum.

The flow we consider leaves invariant the vector space of complex upper Hessenberg matrices and we prove that if the initial condition A_0 is upper Hessenberg and lower triangular (so that the spectrum of A_0 can be readily identified from the diagonal elements of A_0), with complex simple spectrum and with nonzero subdiagonal elements, then $\lim_{t\to+\infty} A(t)$ exists and it is a normal upper Hessenberg matrix, isospectral to A_0 and having nonzero lower codiagonal elements. More in general, we prove that if A_0 is an upper Hessenberg matrix with simple spectrum, then $\lim_{t\to+\infty} A(t)$ exists and it is a normal upper Hessenberg matrix isospectral to A_0. Furthermore if the simple complex spectrum is contained in a line $l \subset \mathbb{C}$, then $\lim_{t\to+\infty} A(t)$ is a normal tridiagonal matrix, isospectral to A_0 and having the nonzero lower codiagonal elements controlled by the nonzero lower codiagonal elements of A_0.

In the following, brackets indicate the usual matrix commutator, $[A, B] := AB - BA$, $A^\dagger$ is the transpose complex conjugate of A and the matrix $[A^\dagger, A]_{du}$ is the matrix equal to $[A^\dagger, A]$ along diagonal and upper triangular entries and zero below diagonal. By lower codiagonal elements or subdiagonal elements of a $n \times n$ matrix A we mean the elements $A_{j+1,j}$, $j = 1, \ldots, n-1$. Moreover, given a differential equation with initial condition $A_0 := A(0)$ we denote with $\Omega(A_0)$ the corresponding ω-limit set (for a definition of ω-limit set and its properties see Chapter 4.2). The main results of the thesis are the following:

Theorem 1.1 *Let A_0 be an upper Hessenberg matrix with simple complex spectrum.*

The flow associated to the following nonlinear system of ODEs

$$\frac{dA}{dt} = [[A^\dagger, A]_{du}, A], \tag{1.0.2}$$

provides an explicit deformation of A_0 to a normal upper Hessenberg matrix A_∞ with the same spectrum, such that the nonzero lower codiagonal elements of A_∞ corresponds to the nonzero codiagonal elements of A_0. Furthermore if A_0 has spectrum contained in a line $l \subset \mathbb{C}$, then its ω-limit set $\Omega(A_0)$ with respect to the flow given by (1.0.2) *is a tridiagonal normal matrix.*

Another interesting result about this flow is that it is the solution of an infinite time horizon optimal control problem. This is given by the following result proved in Chapter 6.3:

Theorem 1.2 *Consider the following deterministic optimal control problem over an infinite time horizon:*

$$\begin{aligned} &\min_U \int_0^{+\infty} \operatorname{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger\right) + \operatorname{trace}(UU^\dagger)\, ds, \\ &\text{subject to } \frac{dA}{dt} = [U, A], \end{aligned} \tag{1.0.3}$$

where $U(t)$ is a sufficiently smooth function taking value in the Lie algebra of upper triangular matrices. Then the optimal value function is given by $V(A) = \operatorname{trace}(AA^\dagger)$ and the optimal feedback is given by $U = [A^\dagger, A]_{du}$, i.e the flow (1.0.2) *is the solution of this infinite time horizon optimal control problem.*

In Chapter 2 we briefly discuss about the matrices, and we introduce matrix group in Chapter 3. In Chapter 4 we discuss about isospectral flow and stability theory. In Chapter 5 we introduce some notations and prove some preliminary results and we prove convergence. In particular, we show that the corresponding ω-limit set is indeed a single matrix of the desired form.

We also analyze how the flow can be used to construct a real even dimensional skew-symmetric tridiagonal matrix with given simple imaginary spectrum and with given signs pattern for the codiagonal elements. This was observed numerically in [1], but a proof was missing.

In Chapter 6 we show that the flow under study is the solution of an infinite time horizon optimal control problem and conclude with some simulations result using MATLAB in Chapter 7.

Chapter 2

Preliminaries

In this chapter we give an introduction to matrices and introduce some notation that will be used throughout this thesis. Let $M_{m\times n}(\mathbb{K})$ is the set of all m by n matrices with the entries in $\mathbb{K}$. $\mathbb{K}$ here is $\mathbb{C}$ or $\mathbb{R}$, except where stated otherwise. The space $M_n(\mathbb{K})$ or $M(n,\mathbb{K})$ is the set of all square matrices of order n. For $A \in M_n(\mathbb{K})$, A_{jk} denotes the (j,k) entry of A.

Definition 2.1 *Let a matrix $A = [A_{jk}] \in M_{m\times n}(\mathbb{K})$ whose entries are in $\mathbb{K}$, the transpose of A, denoted by A^T, is the matrix in $M_{n\times m}(\mathbb{K})$ whose j,k entry is A_{kj}; that is, rows are exchanged with columns and vice versa. The conjugate transpose (also called the adjoint or Hermitian adjoint) of $A = [A_{jk}] \in M_{m\times n}(\mathbb{C})$, is denoted by $A^\dagger$ and defined by $A^\dagger = \bar{A}^T$. The trace of $A = [A_{jk}] \in M_{m\times n}(\mathbb{K})$ is the sum of its main diagonal entries i.e.,* $\mathrm{trace}(A) = A_{11} + \ldots + A_{ll}$, *in which* $l = \min\{m,n\}$. *If $A \in M_n(\mathbb{K})$ then* $\mathrm{trace}(A) = A_{11} + \ldots + A_{nn} = \sum_{l=1}^{n} A_{ll}$.

Definition 2.2 *A matrix $A \in M_n(\mathbb{C})$ is Hermitian matrix if $A^\dagger = A$, skew-Hermitian matrix if $A^\dagger = -A$, unitary matrix if $A^\dagger A = AA^\dagger = \mathbb{I}$ and normal matrix if $A^\dagger A = AA^\dagger$.*

Notice that for any $A \in M_n(\mathbb{C})$, $A+A^\dagger$ is Hermitian and $A-A^\dagger$ is skew-Hermitian. $A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) = H(A) + S(A)$, in which $H(A) = \frac{1}{2}(A + A^\dagger)$ is the Hermitian part of A, $S(A) = \frac{1}{2}(A - A^\dagger)$ is the skew-Hermitian part of A.

If A is Hermitian, then iA is skew-Hermitian and if A is skew-Hermitian, then iA is Hermitian. If A is Hermitian, the main diagonal entries of A are all real and if A is skew-Hermitian, the main diagonal entries of A are all purely imaginary.

Proposition 2.3 *A matrix A is normal if and only if it is unitarily diagonalizable.*

Proof: See [28], pg 246 or [24], pg 133. ∎

Proposition 2.4 *Let $A \in M_n(\mathbb{K})$ is a normal matrix. A has all eigenvalues real iff the matrix A is Hermitian. A has all eigenvalues pure imaginary iff the matrix A is skew-Hermitian.*

Proof: See [21], pg 178 or [24], pg 141. ∎

Definition 2.5 *A matrix $A \in M_n(\mathbb{K})$ is upper Hessenberg matrix if $A_{jk} = 0$ for $j > k+1$. In other words, A is an upper Hessenberg matrix if it is entirely zero below the first subdiagonal entries.*

$$A = \begin{bmatrix} A_{11} & & & & & * \\ A_{21} & A_{22} & & & & \\ & A_{32} & A_{33} & & & \\ & & & & & \\ & & & A_{n-1,n-2} & A_{n-1,n-1} & \\ 0 & & & & A_{n,n-1} & A_{nn} \end{bmatrix}.$$

An upper Hessenberg matrix A is said to be unreduced if all its first subdiagonal entries are nonzero, that is, if $A_{k+1,k} \neq 0$ for all $k = 1, 2, 3, \ \ldots \ , n-1$.

A completely analogous definition holds for a lower Hessenberg matrix.

Definition 2.6 *A matrix $A \in M_n(\mathbb{K})$ that is both upper and lower Hessenberg is called tridiagonal, that is, A is tridiagonal if $A_{jk} = 0$ whenever $|j-k| > 1$.*

$$A = \begin{bmatrix} A_{11} & A_{12} & & & & & 0 \\ A_{21} & A_{22} & A_{23} & & & & \\ & A_{32} & A_{33} & A_{34} & & & \\ & & & & & & \\ & & & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & & & & & A_{n,n-1} & A_{nn} \end{bmatrix}.$$

A matrix $A \in M_n(\mathbb{R})$ is called a Jacobi matrix if it is tridiagonal and symmetric.

Definition 2.7 *Let $A \in M_n(\mathbb{C})$. If a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $x \in \mathbb{C}^n$ satisfy the equation*

$$Ax = \lambda x$$

then λ is called an eigenvalue of A and x is called an eigenvector of A associated with the eigenvalue λ.

Definition 2.8 *A set of the eigenvalues of a matrix A is called its spectrum, and is denoted by $\sigma(A)$ or Spect(A). A matrix A is said to have simple spectrum if all its eigenvalues are distinct.*

Definition 2.9 *Suppose A and B are square matrices of order n over field $\mathbb{K}$. We say A and B are similar if there exists a matrix P such that $B = P^{-1}AP$.*

Proposition 2.10 *If A and B are similar matrices then they have the same eigenvalues with the same geometric multiplicities.*

Proof: $B = P^{-1}AP \iff PBP^{-1} = A$. If $Av = \lambda v$, then $PBP^{-1}v = \lambda v \implies BP^{-1}v = \lambda P^{-1}v$. So, if v is an eigenvector of A, with eigenvalue λ, then $P^{-1}v$ is an eigenvector of B with the same eigenvalue. So, every eigenvalue of A is an eigenvalue of B and since you can interchange the roles of A and B in the previous calculations, every eigenvalue of B is an eigenvalue of A too. Hence, A and B have the same eigenvalues.

Geometrically, in fact, also v and $P^{-1}v$ are the same vector, written in different coordinate systems. Geometrically, in fact, also A and B are matrices associated to the same endomorphism. So, they have the same eigenvalues, eigenvectors and geometric multiplicities. ■

Definition 2.11 *If $A \in M_n(\mathbb{K})$ is similar to a diagonal matrix, then A is said to be diagonalizable.*

Theorem 2.12 *If $A \in M_n(\mathbb{K})$ has n distinct eigenvalues, then A is diagonalizable.*

Proof: See [24], pg 60. ■

$M_n(\mathbb{K})$ is a vector space, it is useful to apply a metric on the vector space. We define metrics called matrix norms that are regular norms with one additional property to the matrix product.

Definition 2.13 *A function $\|\cdot\| : M_n(\mathbb{K}) \to \mathbb{R}$ is a matrix norm if, for all $A, B \in M_n(\mathbb{K})$, it satisfies the following axioms:*
(1) Positivity: $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.
(2) Homogeneity: $\|cA\| = |c|\|A\|$ for all $c \in \mathbb{C}$.
(3) Subadditivity: $\|A + B\| \leq \|A\| + \|B\|$.
(4) Submultiplicativity: $\|AB\| \leq \|A\|\|B\|$.

For example, the Frobenius norm defined for $A \in M_n(\mathbb{C})$ by

$$\|A\|_F = |\mathrm{trace}(AA^\dagger)|^{1/2}$$

where $|\mathrm{trace}(AA^\dagger)|^{1/2} = \sqrt{\sum_{j,k=1}^n |A_{jk}|^2}$. It is easy to see that first three axioms of matrix norm are satisfied by the Frobenious norm. For the fourth axiom, we apply the Cauchy-Schwartz inequality as follows:

$$\begin{aligned}\|AB\|_F^2 = \sum_{j,k}\left|\sum_l A_{jl}B_{lk}\right|^2 &\leq \sum_{j,k}\left(\sum_l |A_{jl}|^2\right)\left(\sum_m |B_{mk}|^2\right) \\ &= \left(\sum_{j,l} |A_{jl}|^2\right)\left(\sum_{m,k} |B_{mk}|^2\right) = \|A\|_F^2\|B\|_F^2.\end{aligned}$$

This proves $\|AB\|_F \leq \|A\|_F\|B\|_F$.

Chapter 3

Matrix Groups

A matrix group (or linear group) is a group G whose elements are invertible matrices over a field. Since invertible matrices represent geometric motions (i.e., linear transformations), matrix groups are useful in geometry. Moreover the flow we are studying is defined on all isospectral manifold which is the orbit under similarity transformation of a matrix Lie group.

3.1 The General Linear Groups

The set $M_n(\mathbb{K})$ is not a group under matrix multiplication because zero matrix does not have multiplicative inverse.

Definition 3.1 *The general linear group over the field* $\mathbb{K}$ *is denoted by* $GL_n(\mathbb{K})$ *and is defined by*

$$GL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) | \det(A) \neq 0\}.$$

$GL_n(\mathbb{K})$ is a group under matrix multiplication. Note that the product of invertible matrices is invertible, inverse of invertible matrices is invertible, matrix multiplication is associative and identity matrix is the identity element. A group G is a matrix group when it is a subgroup of $GL_n(\mathbb{K})$.

Proposition 3.2 *The group* $GL_n(\mathbb{C})$ *is connected for all* $n \geq 1$.

Proof: See [4], pg 13. ■

Theorem 3.3 *$GL_n(\mathbb{C})$ is isomorphic to a subgroup of $GL_{2n}(\mathbb{R})$.*

Proof: See [12], pg 23. ■

3.2 The Unitary Groups

One of the most important subgroups of the general linear groups is the unitary groups.

Definition 3.4 *The unitary group over $\mathbb{C}$ is denoted by $\mathbb{U}(n)$ and defined by*

$$\mathbb{U}(n) = \{U \in GL_n(\mathbb{C}) | UU^\dagger = \mathbb{I}\}.$$

Proposition 3.5 *The group $\mathbb{U}(n)$ is connected for all $n \geq 1$.*

Proof: See [4], pg 14. ■

3.3 Lie Algebras

Notice that $G \subset GL_n(\mathbb{K}) \subset M_n(\mathbb{K}) \cong \mathbb{K}^{n^2} \cong \begin{cases} \mathbb{R}^{n^2} & \text{if } \mathbb{K} = \mathbb{R}. \\ \mathbb{R}^{2n^2} & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$ So we can think of G as a subset of Euclidean space. For example $M_2(\mathbb{C})$ is identified with $\mathbb{R}^8$ via the identification:

$$\begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ z_5 + iz_6 & z_7 + iz_8 \end{pmatrix} \to (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8).$$

Definition 3.6 *Let $G \subset GL_n(\mathbb{K})$, and $H \in G$. The tangent space to G at H is denoted by $T_H G$ and is defined by*

$$T_H G = \{\gamma'(0)|\gamma : (-\epsilon, \epsilon) \to G \ is \ differentiable \ with \ \gamma(0) = H\}.$$

This definition suggests that the tangent space $T_H G$ is the set of all initial velocity vectors of differentiable paths through H in G.

Definition 3.7 *The Lie algebra of a martix group $G \subset GL_n(\mathbb{K})$ is the tangent space to G at $\mathbb{I}$. It is denoted by $\mathfrak{g}(G) := T_{\mathbb{I}}G$.*

Proposition 3.8 *The Lie algebra $\mathfrak{g}$ of a matrix group $G \subset GL_n(\mathbb{K})$ is real subspace of $M_n(\mathbb{K})$.*

Proof: See [12], pg 70. ■

3.4 Manifolds

The function f is called $C^r(X)$ if all r th order partial derivatives exist and are continuous on X. The function f is called smooth on X (or $C^\infty(X)$) if f is $C^r(X)$ for all positive integer r.

Proposition 3.9 *An exponential map $exp : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ is smooth.*

Proof: See [12], pg 100. ■

Proposition 3.10 *exp(A) $\in GL_n(\mathbb{K})$ for all $A \in M_n(\mathbb{K})$. Hence the exponential map $exp : \mathfrak{gl}_n(\mathbb{K}) \to GL_n(\mathbb{K})$.*

Proof: See [12], pg 93. ■

Definition 3.11 *Let $X \subset \mathbb{R}^m$ be open and $F : X \to \mathbb{R}^n$. F is called smooth if for all $x \in X$, there exists a neighborhood V of x in $\mathbb{R}^m$ and a smooth function $\Phi : V \to \mathbb{R}^n$ such that $F \equiv \Phi$ on $X \cap V$.*

Definition 3.12 *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$. X and Y are said to be diffeomorphic if there exists a smooth bijective function $F : X \to Y$ such that $F^{-1} : Y \to X$ is also smooth. The function $F : X \to Y$ is called diffeomorphism.*

A diffeomorphism is a smooth homeomorphism which has a smooth inverse.

Definition 3.13 *Let $X \subset \mathbb{R}^m$. X is called a manifold if for all $x \in X$ there exists a neighborhood V of x which is diffeomorphic to an open subset W of $\mathbb{R}^n$. In this case, X is a manifold of dimension n.*

In other words, a manifold is a set which is locally diffeomorphic to Euclidean space.

Theorem 3.14 *Any matrix group of dimension n is a manifold of dimension n.*

Proof: See [12], pg 106. ■

3.5 Flow of a Vector Field and Lie Derivative

Let M is an n dimensional manifold. For $p \in M$, let $x^j(p)$ denote local coordinates in a subset of M for $j = 1, 2, 3, ..., n$. That is $(x^1(p), x^2(p), ..., x^n(p))$ can be seen as a map from the subset of M to $\mathbb{R}^n$.

A vector field F on M is a map that associates to each point $p \in M$ a tangent vector in T_pM, denoted $F|_p$ or $F(p)$, that is smooth in the sense that in local coordinates $x^1, x^2, ..., x^n$, a vector field has the form $F = \sum a^j(x)\frac{\partial}{\partial x^j}$, we require that the functions $x \mapsto a^j(x)$ be smooth.

The flow ϕ of a vector field F on M is a smooth one parameter group of diffeomorphisms $\phi_t : M \to M$ such that $\phi_0(p) = p$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$ for all $t, s \in \mathbb{R}$ and $\frac{d}{dt}\phi_t(p) = F(\phi_t(p))$, and the map $\phi : \mathbb{R} \times M \to M$ as $(t, p) \mapsto \phi_t(p)$ is smooth.

Let $f : M \to M$ be a smooth map on M then the vector field F on f is written as $F(f)$. For a point $p \in M$, the vector field F for a given function f takes the value $F(f)(p)$. Seen as a function of f, $F(f)(p)$ is a real valued smooth linear map on M. The component F^j of a vector field F in the local coordinate system x^j are given by

$$F(f)(p) = F^j(p)\frac{\partial f}{\partial x^j}(p).$$

Let $\Phi : M \to M$ be a map from M to itself. For given $p \in M$, we can define induced map $d_p\Phi : T_pM \to T_{\Phi(p)}M$ as

$$d_p\Phi(F)(f) = F(f \circ \Phi)(p).$$

In coordinate system

$$d_p\Phi(F) = F^j \frac{\partial \Phi^k}{\partial x^j}\frac{\partial f}{\partial x^k}.$$

Let us define the Lie derivative of a function $f : M \to \mathbb{R}$. The Lie derivative is defined with respect to a vector field F. When acting on a function f it quantifies how much f changes along the flow of F. We should therefore define it using the difference $f(\phi_t(p)) - f(p)$. This measures the change of f in the direction of the flow of F.

Therefore we define the Lie derivative $\mathcal{L}_F f$ of a function $f : M \to \mathbb{R}$ along the vector field F as

$$\mathcal{L}_F f(p) = \lim_{t\to 0}\frac{f(\phi_t(p)) - f(p)}{t} = F(f)$$

for any point $p \in M$ where ϕ_t is the flow of F .

Now define the Lie derivative of a vector field G with respect to a vector field F. When acting on a vector field G it quantifies how much G changes along the flow ϕ_t of F. We could imagine taking the difference between G at p and G at $\phi_t(p)$, and then look at the limit as $t \to 0$. However, this is not well defined since G at $\phi_t(p)$

is in the tangent space at $\phi_t(p)$ to M and G at p is in the tangent space at p to M. In short, they are vectors in two different tangent spaces, hence we cannot subtract them from each other. Therefore, we should find a way to take $G(\phi_t(p))$ and map it into a vector in the tangent space T_pM. This can be done with the induced map

$$d_{\phi_t(p)}\phi_{-t} : T_{\phi_t(p)}M \to T_pM.$$

For any point $p \in M$ and flow ϕ_t of F, the Lie derivative $\mathcal{L}_F G$ of the vector field G along the vector field F is defined as

$$\mathcal{L}_F G(p) = \lim_{t\to 0} \frac{d_{\phi_t(p)}\phi_{-t}(G) - G(p)}{t}.$$

Theorem 3.15 *For any smooth vector fields F and G on a smooth manifold M, $\mathcal{L}_F G = [F, G]$.*

Proof: See [10], pg 333. ∎

3.6 Lie Groups

Definition 3.16 *[31] A Lie group G is a group, which is also a smooth manifold, such that $G \times G \to G, (x, y) \mapsto xy^{-1}$ is a smooth map.*

Example of Lie group is the general linear group $GL_n(\mathbb{C}) = \{U \in M_n(\mathbb{C}) | \det(U) \neq 0\}$ because matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries. Also, the unitary group $\mathbb{U}(n) = \{U \in M_n(\mathbb{C}) | UU^\dagger = \mathbb{I}\}$.

The group $\mathbb{U}(n)$ is compact Lie group, i.e., Lie group which is compact spaces (see for instance [15], pg 5). Also, $GL_n(\mathbb{C})$ and $\mathbb{U}(n)$ are connected.

The tangent space $\mathfrak{g} := T_{\mathbb{I}}G$ of a Lie group G at the identity element $\mathbb{I} \in G$ carries in a natural way the structure of a Lie algebra. The Lie algebras of the Lie group $GL_n(\mathbb{C})$ is $\mathfrak{gl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C})\} = M_n(\mathbb{C})$. Also the Lie algebras of the Lie group $\mathbb{U}(n)$ is skew$(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}) | X^\dagger = -X\}$ (see [31], pg 353 or [4], pg 40). The special linear group $SL_n(\mathbb{C}) = \{T \in M_n(\mathbb{C}) | \det(T) = 1\}$ and the Lie algebra of $SL_n(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) | \text{trace}(X) = 0\}$.

In all of these cases the product structure on the Lie algebra is given by the Lie bracket $[X, Y] = XY - YX$ of matrices X and Y, satisfying the following two properties for all $X, Y, Z \in G$:

1. Antisymmetry: $[X, Y] = -[Y, X]$.

2. Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Theorem 3.17 *Every matrix Lie group is a smooth embedded submanifold of $M_n(\mathbb{C})$ and is thus a Lie group.*

Proof: See [4], pg 22. ■

Theorem 3.18 *The exponential map $exp : \mathfrak{gl}_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is surjective.*

Proof: See [15], pg 87. ■

The following example shows that the exponential map is not surjective in general:

Example 3.19 *The exponential map $exp : \mathfrak{sl}_2(\mathbb{C}) \to SL_2(\mathbb{C})$ is not surjective.*

Proof: Let $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \in SL_2(\mathbb{C})$, claim that there is no matrix $X \in \mathfrak{sl}_2(\mathbb{C})$ with $exp(X) = A$. To prove this, let us consider an arbitrary matrix $X \in \mathfrak{sl}_2(\mathbb{C})$ with trace$(X) = 0$. The eigenvalues of X are in the form λ and $-\lambda$. There are then two possibilities: $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$, $exp(X)$ will have 1 as an eigenvalue. Therefore, $exp(X) \neq A$. If $\lambda \neq 0$, X has distinct eigenvalues and is, therefore, diagonalizable. It follows that $exp(X)$ is also diagonalizable. However, A is not diagonalizable. This shows that $exp(X) \neq A$. ■

Let $\Lambda = \text{diag}(\lambda_1, \ . \ . \ ., \lambda_n)$ with $\lambda_j \neq \lambda_k$ for all $j, k = 1, \ . \ . \ . \ , n$ with $j \neq k$ be a given complex diagonal matrix and let $\mathcal{N}_\Lambda = \{U\Lambda U^\dagger \in M_n(\mathbb{C}) | UU^\dagger = \mathbb{I}\}$ denote the set of all normal matrices $N = U\Lambda U^\dagger$ unitarily equivalent to Λ. Thus $\mathcal{N}_\Lambda$ is the set of all normal matrices with eigenvalues $\{\lambda_1, \lambda_2, \ . \ . \ ., \lambda_n\}$.

The following is the description of the tangent spaces of $\mathcal{N}_\Lambda$.

Lemma 3.20 *The tangent space of* $\mathcal{N}_\Lambda$ *at* $N \in \mathcal{N}_\Lambda$ *is*

$$T_N\mathcal{N}_\Lambda = \{[\Omega, N] | \Omega^\dagger = -\Omega\}.$$

Proof: See [31], pg 49. ■

Chapter 4

Isospectral Flow

4.1 Isospectral Flow

A differential equation on space of matrix $M_n(\mathbb{K})$

$$\dot{A}(t) = F(A(t)), \quad A \in M_n(\mathbb{K}) \tag{4.1.1}$$

is called isospectral, if the eigenvalues of the solutions $A(t)$ are constant in time, i.e., if

$$\text{Spect}\,(A(t)) = \text{Spect}\,(A(0)) \tag{4.1.2}$$

holds for all t and all initial conditions $A(0) \in M_n(\mathbb{K})$. A more restrictive class of isospectral matrix flows are the self-similar flows on $M_n(\mathbb{K})$. These are defined by the property that

$$A(t) = S(t)A(0)S(t)^{-1} \tag{4.1.3}$$

holds for all initial conditions $A(0)$ and times t, and suitable invertible transformations $S(t) \in GL_n(\mathbb{K})$. Thus the Jordan canonical form of the solutions of a self-similar flow does not change in time. It is easily seen that every self-similar flow (4.1.1) has the Lie bracket form

$$\dot{A} = [B, A] = BA - AB \tag{4.1.4}$$

for a suitable matrix-valued function B of the coefficients of A. To see this, differentiating $A(t) = S(t)A(0)S(t)^{-1}$ with respect to t.

$$\dot{A}(t) = \dot{S}(t)A(0)S(t)^{-1} + S(t)A(0)\dot{S}^{-1}(t).$$

Since $\dot{S}(t)^{-1} = -S(t)^{-1}\dot{S}(t)S(t)^{-1}$, we get

$$\begin{aligned}
\dot{A}(t) &= \dot{S}(t)S(t)^{-1}S(t)A(0)S(t)^{-1} - S(t)A(0)S(t)^{-1}\dot{S}(t)S(t)^{-1} \\
&= \dot{S}(t)S(t)^{-1}A(t) - A(t)\dot{S}(t)S(t)^{-1} \\
&= [\dot{S}(t)S(t)^{-1}, A(t)].
\end{aligned}$$

Therefore, every self-similar flow (4.1.1) has the Lie bracket form $\dot{A} = [B, A]$.

Conversely, every differential equation on $M_n(\mathbb{K})$ of the form (4.1.4) is isospectral (see [16], [31]).

Example 4.1 *The simplest example of an isospectral flow is the linear self-similar flow*

$$\dot{A} = [A, B] \tag{4.1.5}$$

for a constant matrix $B \in M_n(\mathbb{K})$.

The solutions are $A(t) = e^{-tB}A(0)e^{tB}$ which shows the self-similar nature of the flow.

Example 4.2 *Another example of an isospectral flow is Brockett's double Lie bracket equation*

$$\dot{A} = [A, [A, B]] = A^2B + BA^2 - 2ABA. \tag{4.1.6}$$

For this equation there is no general solution formula known. See [14] for more example.

Proposition 4.3 *Consider the following differential equation on the space of matrices $M_n(\mathbb{C})$*

$$\dot{A} = [f(t, B), A] \qquad A(0) = A_0 \in M_n(\mathbb{C}) \tag{4.1.7}$$

where $f : [0, \infty) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a matrix-valued function that is locally Lipschitz in the variables on $B \in M_n(\mathbb{C})$ where B is a matrix whose entries are functions of entries of A and continuous in t. Then eigenvalues of $A(t)$ are the same for all t, i.e., $A(t)$ is isospectral.

Proof: Let the eigenvalues of A_0 are $\lambda_1, \lambda_2, \ldots, \lambda_n$. From the result in linear algebra if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k for any integer k.

First of all, check that $\frac{d}{dt}\left(\text{trace}(A^k)\right) = 0$ if (4.1.7) is satisfied.

$$\begin{aligned}
\frac{d}{dt}\left(\text{trace}(A^k)\right) &= \frac{d}{dt}\left(\text{trace}(A \cdot A \cdot A \cdots A)\right) \\
&= \text{trace}(\dot{A} \cdot A^{k-1}) + \text{trace}(A \cdot \dot{A} \cdot A^{k-2}) + \cdots + \text{trace}(A^{k-1} \cdot \dot{A}) \\
&= \text{trace}(\dot{A} \cdot A^{k-1}) + \text{trace}(\dot{A} \cdot A^{k-1}) + \cdots + \text{trace}(\dot{A} \cdot A^{k-1})
\end{aligned}$$

using the property $\text{trace}(A \cdot B) = \text{trace}(B \cdot A)$. Hence,

$$\frac{d}{dt}\left(\text{trace}(A^k)\right) = k \cdot \text{trace}(\dot{A} \cdot A^{k-1}).$$

Now using the flow (4.1.7) we get

$$\frac{d}{dt}\left(\text{trace}(A^k)\right) = k \cdot \text{trace}([f(t, B), A]A^{k-1}) = k \cdot \text{trace}(f(t, B)[A, A^{k-1}]) = 0$$

since $\text{trace}([A, B]C) = \text{trace}(A[B, C])$ and $[A, A^{k-1}] = 0$.

This implies that $\frac{d}{dt}\left(\sum_{j=1}^{n} \lambda_j^k\right) = 0$. This gives $\left(\sum_{j=1}^{n} \lambda_j^k\right)$ is constant, does not depend on t. ∎

See the detailed proof of Proposition 4.3 in Appendix A.

4.2 Stability Theory

In this section we review Lyapunov stability theory. Consider a dynamical system

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \qquad x \in \mathbb{R}^n. \tag{4.2.1}$$

We will assume that $f(t, x)$ satisfies the standard conditions for the existence and uniqueness of solutions.

Definition 4.4 *A point $x^* \in \mathbb{R}^n$ is an equilibrium point of* (4.2.1) *if $f(t, x^*) \equiv 0$.*

Definition 4.5 *An equilibrium point x^* of* (4.2.1) *is locally stable if all solutions which start in a neighborhood of x^* remain in a neighborhood of x^* for all time.*

Definition 4.6 *The equilibrium point x^* of* (4.2.1) *is said to be locally asymptotically stable if x^* is locally stable and, furthermore, all solutions starting near x^* tend towards x^* as $t \to \infty$.*

By shifting the origin of the system, we may assume that the equilibrium point of interest occurs at $x^* = 0$.

Definition 4.7 *The equilibrium point $x^* = 0$ of* (4.2.1) *is said to be stable (in the sense of Lyapunov) at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta > 0$ depending on t_0 and ϵ, such that*

$$||x(t_0)|| < \delta \quad \Longrightarrow ||x(t)|| < \epsilon, \;\; \forall t \geq t_0.$$

Definition 4.8 *The equilibrium point $x^* = 0$ of* (4.2.1) *is said to be asymptotically stable at $t = t_0$ if $x^* = 0$ is stable and $x^* = 0$ is locally attractive, i.e., there exists a*

$\delta > 0$ *depending on* t_0*, such that*

$$||x(t_0)|| < \delta \quad \Longrightarrow \lim_{t \to t_0} ||x(t)|| = 0.$$

Above definitions are local definitions; they describe the behavior of a system near an equilibrium point. We say an equilibrium point x^* is **globally** stable if it is stable for all initial conditions $x_0 \in \mathbb{R}^n$.

Definition 4.9 *The equilibrium point* $x^* = 0$ *of* (4.2.1) *is said to be an exponentially stable at* $t = t_0$ *if there exist constants* $M, \alpha > 0$ *and* $\epsilon > 0$ *such that*

$$||x(t)|| \leq Me^{-\alpha(t-t_0)}||x(t_0)||$$

for all $||x(t_0)|| \leq \epsilon$ *and* $t \geq t_0$*.*

The largest constant α in the definition is called the *rate of convergence.* Exponential stability is a strong form of stability; in particular, it implies uniform, asymptotic stability.

A system is *globally exponentially stable* if the bound in above definition holds for all initial conditions $x_0 \in \mathbb{R}^n$.

4.2.1 Lyapunov Stability for Autonomous Systems

When (4.2.1) does not depend on time t explicitly, i.e.,

$$\dot{x} = f(x) \qquad x(t_0) = x_0 \tag{4.2.2}$$

for continuous time then the system becomes an autonomous system.

Theorem 4.10 *Lyapunov Theorem*

For autonomous systems, let $D \subset \mathbb{R}^n$ be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function $V : D \to \mathbb{R}$ such that

$$\dot{V} = \frac{\partial V(x)}{\partial x}\frac{dx}{dt} = \frac{\partial V(x)}{\partial x} f(x) \leq 0$$

on D then, the equilibrium point 0 is stable.

Moreover, if $\dot{V}$ is negative definite then the equilibrium is asymptotically stable.

In addition, if $D = \mathbb{R}^n$ and V is radially unbounded or proper, then, the origin is globally asymptotically stable.

Proof: See [8], pg 114 or [25]. ∎

For autonomous systems, when $\dot{V}$ in the above theorem is only negative semi-definite, asymptotic stability may still be obtained by applying the LaSalle's invariance principle.

4.2.2 Lyapunov Stability for Nonautonomous Systems

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation (4.2.1).

Definition 4.11 *Let B_ϵ be a ball of size ϵ around the origin, $B_\epsilon = \{x \in \mathbb{R}^n : ||x|| < \epsilon\}$. A continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a locally positive definite function(lpdf) if for some $\epsilon > 0$ and some continuous, strictly increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}$,*

$$V(t, 0) = 0 \quad \text{and} \quad V(t, x) \geq \alpha(||x||) \qquad \forall x \in B_\epsilon, \quad \forall t \geq 0.$$

Definition 4.12 *A continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a positive definite function(pdf) if is locally positive definite function and $\alpha(p) \to \infty$ as $t \to \infty$.*

Definition 4.13 *A continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a decrescent function* *if for some $\epsilon > 0$ and some continuous, strictly increasing function $\beta : \mathbb{R}^+ \to \mathbb{R}$,*

$$V(t,x) \leq \beta(||x||) \qquad \forall x \in B_\epsilon, \quad \forall t \geq 0.$$

The following theorem states that when $V(t,x)$ is a lpdf and $\dot{V}(t,x) \leq 0$ then we can conclude stability of the equilibrium point.

Theorem 4.14 *Basic Theorem of Lyapunov*

Let $V(t,x)$ be a nonnegative function with derivative $\dot{V}$ along the trajectories of the system.

- *If $V(t,x)$ is lpdf and $\dot{V}(t,x) \leq 0$ locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).*
- *If $V(t,x)$ is lpdf and decrescent, and $\dot{V}(t,x) \leq 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).*
- *If $V(t,x)$ is lpdf and decrescent, and $-\dot{V}(t,x)$ is lpdf, then the origin of the system is uniformly locally asymptotically stable.*
- *If $V(t,x)$ is pdf and decrescent, and $-\dot{V}(t,x)$ is pdf, then the origin of the system is globally uniformly asymptotically stable.*

Proof: See [8], pg 150. ■

Above theorem gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(t,x)$.

Theorem 4.15 *Exponential stability theorem*

$x^* = 0$ is an exponentially stable equilibrium point of $\dot{x} = f(t,x)$ if and only if there exists an $\epsilon > 0$ and a function $V(t,x)$ which satisfies

$$M_1||x||^2 \leq V(t,x) \leq M_2||x||^2$$

$$\dot{V}|_{\dot{x}=f(t,x)} \leq -M_3||x||^2$$

$$||\frac{\partial V}{\partial x}(t,x)|| \leq M_4||x||$$

for some positive constants M_1*,* M_2*,* M_3*,* M_4*, and* $||x|| \leq \epsilon$*.*

Proof: See [27], pg 28. ■

4.2.3 Linearization Method

In this section, we are going to discuss about one of the most useful results in Lyapunov stability theory, namely the linearization method. This method enables us to draw conclusions about a nonlinear system by studying the behaviour of a linear system. First we define the concept of linearizing a nonlinear system around an equilibrium. Consider the time invariant nonlinear system:

$$\dot{x} = f(x) \tag{4.2.3}$$

suppose $f(0) = 0$, so that 0 is an equilibrium of the system (4.2.3), and suppose also that f is continuously differentiable. Let A be the Jacobian of f evaluated at $x = 0$, i.e.,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

By the definition of the Jacobian, it follows that if we define $f(x) = Ax + g(x)$ then,

$$\lim_{||x|| \to 0} \frac{||g(x)||}{||x||} = 0.$$

The system

$$\dot{\xi} = A\xi \tag{4.2.4}$$

is the linearization of (4.2.3) around the equilibrium 0.

Now consider time-varying system

$$\dot{x} = f(t, x) \tag{4.2.5}$$

with $f(t, 0) = 0$, for all $t \geq 0$ and f is a continuously differentiable function. Define

$$A(t) = \left. \frac{\partial f(t, x)}{\partial x} \right|_{x=0}$$

and $f(t, x) = A(t)x + g(t, x)$ then,

$$\lim_{||x|| \to 0} \frac{||g(t, x)||}{||x||} = 0.$$

However, it may not be true that

$$\lim_{||x|| \to 0} \sup_{t \geq 0} \frac{||g(t, x)||}{||x||} = 0. \tag{4.2.6}$$

The system

$$\dot{\xi} = A(t)\xi \tag{4.2.7}$$

is the linearization of (4.2.5) around the equilibrium 0 provided (4.2.6) holds.

Here, we present the main stability theorem of the linearization method. The result is stated for a general time-varying system.

Theorem 4.16 *Consider the system* (4.2.5). *Suppose that $f(t, 0) = 0$, for all $t \geq 0$ and that $f(\cdot)$ is continuously differentiable. Define $A(t)$, $g(t, x)$ as above, and assume that*

$$\lim_{||x|| \to 0} \sup_{t \geq 0} \frac{||g(t, x)||}{||x||} = 0.$$

holds, and $A(\cdot)$ is bounded. If 0 is an exponentially stable equilibrium of the linear system (4.2.7) *then it is also an exponentially stable equilibrium of the system* (4.2.5).

Proof: See [17], pg 212. ■

4.2.4 Limit Set

A limit set is the state a dynamical system reaches after an infinite amount of time has passed, by either going forward or backwards in time.

Definition 4.17 *[5] Suppose that ϕ_t is a flow on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. A point x in $\mathbb{R}^n$ is called an omega limit point (ω-limit point) of the orbit through p if there is a sequence of numbers $t_1 \leq t_2 \leq t_3 \leq \ldots.$ such that* $\lim_{j\to\infty} t_j = \infty$ and $\lim_{j\to\infty} \phi_{t_j}(p) = x$. *The collection of all such omega limit points is denoted $\Omega(p)$ and is called the omega limit set (ω-limit set) of p.*

Definition 4.18 *[5] The orbit of the point p with respect to the flow ϕ_t is called forward complete if $t \to \phi_t(p)$ is defined for all $t \geq 0$. Also, in this case, the set $\{\phi_t(p) : t \geq 0\}$ is called the forward orbit of the point p. The orbit is called backward complete if $t \to \phi_t(p)$ is defined for all $t \leq 0$ and the backward orbit is $\{\phi_t(p) : t \leq 0\}$.*

4.2.5 LaSalle's Invariance Principle

LaSalle's theorem enables one to conclude asymptotic stability of an equilibrium point even when $-\dot{V}(x,t)$ is not locally positive definite. However, it applies only to autonomous or periodic systems.

We denote the solution trajectories of the autonomous system

$$\dot{x} = f(x) \qquad x(t_0) = x_0 \tag{4.2.8}$$

as $x(t)$, which is the solution of the equation (4.2.8) at time t starting from x_0 at t_0.

Definition 4.19 *The set $M \subset \mathbb{R}^n$ is said to be an invariant set with respect to* (4.2.8) *if for all $x_0 \in M$, we have $x(t) \in M$, $\forall t \in \mathbb{R}$ and a positively invariant set if for all*

$x_0 \in M$, we have $x(t) \in M$, $\forall t \geq 0$.

Proposition 4.20 *The ω-limit set of a point is closed and invariant.*

Proof: See [5], pg 92. ∎

Theorem 4.21 *LaSalle's invariance principle*

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$ we have $\dot{V}(x) \leq 0$. Define

$$S = \{x \in \Omega_c : \dot{V}(x) = 0\}.$$

As $t \to \infty$, the trajectory tends to the largest invariant set inside S; i.e., its ω-limit set is contained inside the largest invariant set in S. In particular, if S contains no invariant sets other than $x \equiv 0$, then the origin is asymptotically stable.

Proof: See [8], pg 128. ∎

4.2.6 Strengthening LaSalle's Invariance Principle

Assumptions 4.22 *[2] Let us assume the following:*

- *A Riemannian manifold M of class C^2 with metric g on which a locally Lipschitz continuous vector field*
$$\dot{x} = f(x) \tag{4.2.9}$$
is given.

- *We consider a Cauchy problem for (4.2.9) with initial value $x(0)$ is such that the corresponding solution $x(t, x(0))$ is bounded.*

- *We assume that the ω-limit set $\Omega(x(0))$, which is a compact and connected set is contained in a closed embedded submanifold $S \subset M$. Equivalently, we assume that S is attracting for the solution of (4.2.9) starting at $x(0)$.*

- *Call O an open tubular neighborhood of S in M. We assume that there exists a real valued C^1 function $W : O \to \mathbb{R}$ and such that $\dot{W}(x) \geq 0$ on S (or $\dot{W}(x) \leq 0$ on S), where $\dot{W}$ is the derivative of $W(x)$ along the flow (Lie derivative). Moreover, let $E := \{x \in S : \dot{W}(x) = 0\}$ so that $\dot{W}(x) > 0$ on $S \backslash E$ (or $\dot{W}(x) < 0$ on $S \backslash E$).*

Definition 4.23 *[2] Let $\{E_j\}_{j \in I}$ be the connected components of E. Given a function W as in the Assumptions 4.22, we say that the components $\{E_j\}_{j \in I}$ are contained in W if each E_j lies in a level set of W, and the subset $\{W(E_j)\}_{j \in I} \subset \mathbb{R}$ has at most a finite number of accumulation points in $\mathbb{R}$.*

Theorem 4.24 *[2] Assume the Assumptions 4.22 hold. If the components $\{E_j\}_{j \in I}$ are contained in W according to Definition 4.23, then $\Omega(x(0)) \subset E_j$ for a unique $j \in I$.*

This theorem provides a separation argument for detecting ω-limit sets. Whenever the images through W of the connected components of $E := \{x \in S : \dot{W}(x) = 0\}$ are sufficiently separated $\Omega(x(0))$ must be contained in one and only one connected component.

Remark 4.25 *[2] Assume that a positive semidefinite Lyapunov function $V : M \to \mathbb{R}$ is given such that $\dot{V}(x) = 0$ on a submanifold S and $\dot{V}(x) = 0$ on $M \backslash S$. If S is invariant under* (4.2.9)*, then LaSalle's invariance principle claims that the $\Omega(x(0)) \subset S$.*

Under this circumstances, even if another function $W : O \to \mathbb{R}$ is given such that $\dot{W}(x) \geq 0$ on S and W is bounded from above on S, it is not possible to conclude in general that a solution of (4.2.9) starting outside S will converge to a connected component of the set $E = \{x \in S : \dot{W}(x) = 0\}$.

However, Theorem 4.24 shows that this is the case if the connected components of E are contained in level sets of W according to the Definition 4.23, even if $\dot{W}(x) \geq 0$ is indefinite in the tubular neighborhood. If the connected components of E are not contained in level sets of W, then the result might fail.

Notice that it is not necessary that S is invariant, neither that W is positive semidefinite on S to hold Theorem 4.24.

Chapter 5

Main Result

5.1 Preliminary Results

Since we work over the complex field, all the matrices are assumed to have complex entries, unless specified otherwise.

Given a matrix A, we denote with A_d the matrix equal to A along diagonal entries and zero everywhere else, with A_u, the matrix equal to A along strictly upper diagonal elements and zero everywhere else, and with A_l the matrix equal to A along strictly lower diagonal elements and zero everywhere else. Moreover, we combine this notation in the following way: the notation A_{du} stands for a matrix equal to A along diagonal and strictly upper diagonal elements and zero everywhere else and similarly for A_{dl}.

Denote with H^+ the vector space of upper Hessenberg matrices. Recall that a matrix $A \in H^+$ if and only if $A_{jk} = 0$ for $j > k + 1$. In other terms, A is an upper Hessenberg matrix if it is entirely zero below the first subdiagonal (also called lower codiagonal). A completely analogous definition holds for a lower Hessenberg matrix. H^+ is not a Lie algebra under matrix commutator.

We will also denote with $\mathcal{G}_0$ the connected component containing the identity of the Lie group of invertible upper triangular matrices with determinant equal to one and with $\mathfrak{g}_0$ the corresponding Lie algebra, consisting of upper triangular matrices with trace equal to zero. Notice that matrices in $\mathcal{G}_0$ have necessarily nonzero diagonal entries.

Given a matrix A_0 with simple spectrum Λ, we indicate with $\mathcal{N}_\Lambda$ the compact manifold consisting of all normal matrices isospectral to A_0 and with $\mathcal{T}_\Lambda$ the subset of $\mathcal{N}_\Lambda$ consisting of all normal tridiagonal matrices isospectral to A_0.

Lemma 5.1 *$\mathcal{N}_\Lambda$ is a closed (i.e., compact and without boundary) embedded submanifold of $M_n(\mathbb{C})$.*

Proof: Indicating with Λ also a diagonal matrix having spectrum Λ, we have that $\mathcal{N}_\Lambda = \bigcup_{U \in \mathbb{U}(n)} U^\dagger \Lambda U$ where $\mathbb{U}(n)$ is the unitary group. Since the map $\pi : \mathbb{U}(n) \to \mathcal{N}_\Lambda$ given by $U \mapsto U^\dagger \Lambda U$ is continuous and surjective by definition, and since $\mathbb{U}(n)$ is compact, we have that $\mathcal{N}_\Lambda$ is compact too. Since the similarity action of $\mathbb{U}(n)$ on $M_n(\mathbb{C})$ is smooth and $\mathbb{U}(n)$ is a compact Lie group, the action is actually proper and smooth. For proper smooth actions, it is known that orbits are embedded closed submanifolds of the manifold on which the Lie group acts (see for instance [32], Lemma B.19 in Appendix B). In particular, $\mathcal{N}_\Lambda$ is an embedded closed submanifold of $M_n(\mathbb{C})$. ■

In this thesis we study the following nonlinear system of ODEs in Lax form:

$$\frac{dA}{dt} = [[A^\dagger, A]_{du}, A] \tag{5.1.1}$$

and we show that we can use (5.1.1) to obtain an explicit deformation from complex upper Hessenberg matrices to normal upper Hessenberg matrices.

Since (5.1.1) is a polynomial vector field on the vector space of all $n \times n$ matrices with complex coefficients, the classical theorem of existence and uniqueness implies that the corresponding Cauchy problem has always a unique (local) solution (see for instance [29]).

It is also well known that the flow associated to (5.1.1) is isospectral , meaning that the eigenvalues of $A(t)$ are first integrals (see for instance [22]).

First we observe the following about the invariance properties of the flow:

Lemma 5.2 *If $A(t)$ satisfies* (5.1.1) *then $\tilde{A}(t) = A(t) + c\mathbb{I}$ satisfies the same equation for all $c \in \mathbb{C}$, where $\mathbb{I}$ is the identity matrix. Likewise, if $A(t)$ satisfies* (5.1.1) *then $\tilde{A}(t) = e^{i\theta}A(t)$ satisfies the same equation for all $\theta \in \mathbb{R}$.*

Proof: Differentiating $\tilde{A}(t) = A(t) + c\mathbb{I}$, we get $\dot{\tilde{A}}(t) = \dot{A}(t)$. Using the flow (5.1.1)

$$\dot{\tilde{A}} = \left[[A^\dagger, A]_{du}, A\right],$$

add a zero matrix $\left[[A^\dagger, A]_{du}, c\mathbb{I}\right]$ on the right side and simplify, we get

$$\dot{\tilde{A}} = \left[[A^\dagger, A]_{du}, A\right] + \left[[A^\dagger, A]_{du}, c\mathbb{I}\right] = \left[[A^\dagger, A]_{du}, A + c\mathbb{I}\right] = \left[[A^\dagger, A]_{du}, \tilde{A}\right].$$

Note that $[A^\dagger, A] = [A^\dagger + \bar{c}\mathbb{I}, A + c\mathbb{I}] = [(A + c\mathbb{I})^\dagger, A + c\mathbb{I}] = [\tilde{A}^\dagger, \tilde{A}]$. Hence, $\dot{\tilde{A}} = \left[[\tilde{A}^\dagger, \tilde{A}]_{du}, \tilde{A}\right]$.

Similarly, for the second part of the lemma, differentiating $\tilde{A}(t) = e^{i\theta}A(t)$, we get $\dot{\tilde{A}}(t) = e^{i\theta}\dot{A}(t)$. Using the flow (5.1.1)

$$\dot{\tilde{A}} = e^{i\theta}\left[[A^\dagger, A]_{du}, A\right] = \left[[A^\dagger, A]_{du}, e^{i\theta}A\right] = e^{-i\theta}e^{i\theta}\left[[A^\dagger, A]_{du}, \tilde{A}\right] = \left[[e^{-i\theta}A^\dagger, e^{i\theta}A]_{du}, \tilde{A}\right].$$

Hence, $\dot{\tilde{A}} = \left[[\tilde{A}^\dagger, \tilde{A}]_{du}, \tilde{A}\right]$. ■

Some of the relevant properties of (5.1.1) are given by the following:

Lemma 5.3 *For the vector field* (5.1.1) *the following properties hold:*

1. *H^+ is an invariant vector space, namely if the initial condition $A(0)$ is an upper Hessenberg matrix, $A(t)$ will remain upper Hessenberg for all times for which the solution is defined.*

2. *The flow is forward complete, so $A(t)$ exists for all $t \geq 0$. More precisely we have $A(t) \in \mathcal{B}(0, R]$, for all $t \geq 0$ where $\mathcal{B}(0, R]$ is the closed ball in $M_n(\mathbb{C})$ centered at zero, with radius $R := \sqrt{\mathrm{trace}(A(0)A(0)^\dagger)}$.*

3. *Equilibria are normal matrices within the vector space of upper Hessenberg matrices.*

4. *If $A_0 := A(0) \in H^+$ and with simple spectrum Λ, then the ω-limit set $\Omega(A(0))$ is contained in the compact manifold of normal matrices isospectral to A_0, i.e., $\Omega(A(0)) \subset \mathcal{N}_\Lambda$ (more precisely $\Omega(A(0)) \subset \mathcal{N}_\Lambda \cap H^+$).*

5. *If $A_0 := A(0) \in H^+$ and with simple spectrum Λ contained in a line $l \subset \mathbb{C}$, then the ω-limit set $\Omega(A(0))$ is contained in the space of tridiagonal matrices, i.e., $\Omega(A(0)) \subset \mathcal{T}_\Lambda$.*

Proof: The first point is a straightforward calculation. It is sufficient to observe that the product of the upper triangular matrix $[A^\dagger, A]_{du}$ and an upper Hessenberg matrix A is necessarily upper Hessenberg.

To prove the second claim, we show that the Frobenius norm of A is actually monotonically decreasing along (5.1.1), as long as $[A(t)^\dagger, A(t)] \neq 0$. Consider the positive definite quadratic form:

$$V(A) := \|A\|_F^2 := \mathrm{trace}(AA^\dagger). \tag{5.1.2}$$

Differentiating (5.1.2) with respect to (5.1.1) and using $\mathrm{trace}(AB) = \mathrm{trace}(BA)$,

$\mathrm{Re}(\mathrm{trace}(A^\dagger)) = \mathrm{Re}(\mathrm{trace}(A))$ we get:

$$\begin{aligned}
\dot{V}(A) &= \mathrm{trace}(\dot{A}A^\dagger) + \mathrm{trace}(A\dot{A}^\dagger) = \mathrm{trace}(\dot{A}A^\dagger) + \mathrm{trace}\left(((A\dot{A}^\dagger)^\dagger)^\dagger\right) = \\
&= \mathrm{trace}(\dot{A}A^\dagger) + \mathrm{trace}\left((\dot{A}A^\dagger)^\dagger\right) = \mathrm{trace}(\dot{A}A^\dagger) + \overline{\mathrm{trace}\left((\dot{A}A^\dagger)\right)} = \\
&\text{(because } \mathrm{trace}(A^\dagger) = \overline{\mathrm{trace}(A)}) \\
&= 2\,\mathrm{Re}\left(\mathrm{trace}(\dot{A}A^\dagger)\right) = 2\,\mathrm{Re}\left(\mathrm{trace}([[A^\dagger, A]_{du}, A]A^\dagger)\right) = \\
&= -2\,\mathrm{Re}\left(\mathrm{trace}([A, [A^\dagger, A]_{du}]A^\dagger)\right).
\end{aligned}$$

Using $\mathrm{trace}([A,B]C) = -\mathrm{trace}([A,C]B) = \mathrm{trace}([C,A]B)$, from the last equality we obtain:

$$\dot{V}(A) = -2\,\mathrm{Re}\left(\mathrm{trace}([A^\dagger, A][A^\dagger, A]_{du})\right). \tag{5.1.3}$$

Since $[A^\dagger, A]^\dagger = [A^\dagger, A]$, we have that $[A^\dagger, A]$ is a Hermitian matrix. We can use the following fact to prove that $\dot{V}(A)$ is negative as long as $[A(t)^\dagger, A(t)] \neq 0$. For a Hermitian matrix X, using the decomposition $X = (X_{du})^\dagger + X_u$ the following holds:

$$XX_{du} = \left((X_{du})^\dagger + X_u\right) X_{du} = (X_{du})^\dagger X_{du} + X_u X_{du} \quad \text{and} \quad \mathrm{trace}(X_u X_{du}) = 0.$$

This implies that

$$\dot{V}(A) = -2\,\mathrm{Re}\,\mathrm{trace}([A^\dagger, A][A^\dagger, A]_{du}) = -2\,\mathrm{Re}\,\mathrm{trace}\left(([A^\dagger, A]_{du})^\dagger [A^\dagger, A]_{du}\right)$$

$$= -2\|[A^\dagger, A]_{du}^\dagger\|_F^2 = -2\|[A^\dagger, A]_{du}\|_F^2 \leq 0. \tag{5.1.4}$$

Therefore the Lie derivative of $V(A)$ along the vector field (5.1.1) is negative definite, as long as $[A^\dagger, A]_{du}$ is nonzero, or equivalently as long as $[A^\dagger, A]$ is nonzero. In particular the Frobenius norm of A remains bounded and so the flow (5.1.1) is forward complete, since $V(A)$ is a proper function, i.e., its inverse image sends compact sets to compact sets.

Since $\dot{V}(A) \leq 0$, $V(A)$ is non increasing along the vector field (5.1.1). Therefore

$$\|A\|_F^2 = V(A) \leq V\left(A(0)\right) = \text{trace}\left(A(0)A(0)^\dagger\right)$$

and thus $A(t) \in \mathcal{B}(0, R]$ with $R = \sqrt{\text{trace}\left(A(0)A(0)^\dagger\right)}$ follows.

As for the third point, since $\dot{V}(A) = 2\,\text{Re}\left(\text{trace}(\dot{A}A^\dagger)\right)$ and $\dot{A} = 0$ at an equilibrium, we observe that $\dot{V}(A) = 0$ at an equilibrium, so from (5.1.4) we get $[A^\dagger, A]_{du} = 0$. This implies that $\left([A^\dagger, A]\right)_{jk} = 0$ for all $k \geq j$. Since $[A^\dagger, A]$ is Hermitian, this implies $[A^\dagger, A] = 0$ identically. Therefore A is normal. Vice versa, if $\dot{V}(A) = 0$, then it is immediate that A is necessarily an equilibrium.

For the proof of fourth point, by the second point we know that $A(t)$ stays in a closed ball and $\|A(t)\|_F$ is strictly decreasing as long as $A(t)$ is not an equilibrium. By definition of ω-limit set, we can extract a sequence of times $\{t_n\}$ with $t_n \nearrow \infty$, such that $A(t_n)$ converges to a matrix $A_\infty \in \Omega(A(0))$, with $\|A_\infty\|_F = \inf_{n\in\mathbb{N}} \|A(t_n)\|_F$. Then A_∞ is an equilibrium. To prove this, assume for the sake of contradiction that A_∞ is not an equilibrium, and consider the flow ϕ_t of

$$\frac{dA}{dt} = [[A^\dagger, A]_{du}, A].$$

Let $\epsilon > 0$ and sufficiently small, then since $\|A(t)\|_F$ is strictly decreasing, we get

$$\|\phi_\epsilon(A_\infty)\|_F < \|A_\infty\|_F, \tag{5.1.5}$$

since we assumed A_∞ is not an equilibrium. On the other hand, using the group property of the flow ϕ_t it is immediate to see that $\phi_\epsilon(A_\infty)$ is the limit as $n \to +\infty$ of $\phi_{t_n+\epsilon}(A(0))$. However, since $\|\phi_{t_n+\epsilon}(A(0))\|_F > \|A_\infty\|_F$, taking the limit for $n \to +\infty$

we get

$$\|\phi_\epsilon(A_\infty)\|_F \geq \|A_\infty\|_F$$

which contradicts (5.1.5).

Therefore, any accumulation point $A_\infty \in \Omega(A(0))$ is an equilibrium, hence normal by the previous point. Moreover, since the flow preserves upper Hessenberg matrices and $A(0)$ is upper Hessenberg, then any accumulation point is actually given by a *normal upper Hessenberg matrix.* In particular, the ω-limit set $\Omega(A(0))$ is contained in $\mathcal{N}_\Lambda$, the compact embedded manifold consisting of all normal matrices isospectral to $A(0)$.

Finally, for the fifth point we first remark the following. If $A(0)$ has simple real spectrum then $A_\infty \in \Omega(A(0))$ is a Hermitian matrix. Analogously if $A(0)$ has simple imaginary spectrum then $A_\infty \in \Omega(A(0))$ is skew-Hermitian (see Proposition 2.4).

If the spectrum of $A(0)$ is contained in a line $l \subset \mathbb{C}$, then using the invariance property of the flow given in Lemma 5.2, and performing a translation and a rotation we can assume that the spectrum of the transformed initial condition $\tilde{A}(0)$ is contained in the real line. But if $\tilde{A}(0)$ has spectrum contained in the real axis, then $\tilde{A}_\infty$ is Hermitian and since it has to be upper Hessenberg, then it is tridiagonal. Again using the invariance property of the flow we have that $\tilde{A}_\infty = e^{i\theta}(A_\infty + c\mathbb{I})$ which implies that also $A_\infty = e^{-i\theta}\tilde{A}_\infty - c\mathbb{I}$ is tridiagonal. Hence, $\Omega(A(0)) \subset \mathcal{T}_\Lambda$. ■

Let us remark the following. First of all, the fifth point of Lemma 5.3 is not true in the case of general simple spectrum(i.e., if the spectrum is not contained in a line). An example of this fact will be given in Chapter 7, where we consider an initial condition A_0 with spectrum contained in a circle.

Secondly, from the relation $A_\infty = e^{-i\theta}\tilde{A}_\infty - c\mathbb{I}$, using the fact that $\tilde{A}_\infty$ is tridiagonal Hermitian, we can get some information about the entries of A_∞ knowing c and θ, i.e., the location of the line l in which the spectrum of $A(0)$ is contained.

In particular we can obtain the following information about entries of A_∞:

Proposition 5.4 *Given $A(0)$ with simple spectrum lying on a line $l \subset \mathbb{C}$, the co-diagonal elements of matrices A_∞ in $\Omega(A(0))$ satisfy the relation $e^{i\theta}\left(A_\infty\right)_{j+1,j} = e^{-i\theta}\left(\bar{A}_\infty\right)_{j,j+1}$, where θ is the angle formed by the line l with the positive real axis in the complex plane. Moreover $|\left(A_\infty\right)_{j+1,j}| = |\left(A_\infty\right)_{j,j+1}|$.*

Proof: Using Lemma 5.2, we perform a transformation on $A(0)$ to get $\tilde{A}(0)$ with simple spectrum lying on $\mathbb{R}$ or $i\mathbb{R}$.

Without loss of generality, let us consider the case $\tilde{A}(0)$ has simple spectrum lying on $\mathbb{R}$, then the elements in $\Omega(\tilde{A}(0))$ are Hermitian. Therefore,

$$\tilde{A}_\infty = \tilde{A}_\infty^\dagger$$

$$\Longrightarrow\ e^{i\theta}\left(A_\infty + c\mathbb{I}\right) = \left(e^{i\theta}(A_\infty + c\mathbb{I})\right)^\dagger = e^{-i\theta}\left(A_\infty^\dagger + \bar{c}\mathbb{I}\right)$$

$$\Longrightarrow\ e^{i\theta}A_\infty + ce^{i\theta}\mathbb{I} = e^{-i\theta}A_\infty^\dagger + \bar{c}e^{-i\theta}\mathbb{I}.$$

Since $ce^{i\theta}\mathbb{I}$ and $\bar{c}e^{-i\theta}\mathbb{I}$ are diagonal matrices, their presence do not effect on the co-diagonal elements. Therefore, taking the $(j+1, j)$ element on both sides we get:

$$\left(e^{i\theta}A_\infty\right)_{j+1,j} = \left(e^{-i\theta}A_\infty^\dagger\right)_{j+1,j}$$

$$\Longrightarrow\ e^{i\theta}\left(A_\infty\right)_{j+1,j} = e^{-i\theta}\left(A_\infty^\dagger\right)_{j+1,j}$$

$$\Longrightarrow\ e^{i\theta}\left(A_\infty\right)_{j+1,j} = e^{-i\theta}\left(\bar{A}_\infty\right)_{j,j+1}.$$

Therefore,

$$|e^{i\theta}\left(A_\infty\right)_{j+1,j}| = |e^{-i\theta}\left(\bar{A}_\infty\right)_{j,j+1}|$$

$$\Longrightarrow\ |e^{i\theta}||\left(A_\infty\right)_{j+1,j}| = |e^{-i\theta}||\left(\bar{A}_\infty\right)_{j,j+1}|$$

$$\Longrightarrow\ |\left(A_\infty\right)_{j+1,j}| = |\left(\bar{A}_\infty\right)_{j,j+1}|.$$

Hence, $|\,(A_\infty)_{j+1,j}\,| = |\,(A_\infty)_{j,j+1}\,|$, since $|z| = |\bar{z}|$. ■

The following lemma provides some further information about the evolution of subdiagonal elements of an upper Hessenberg matrix A_0 subject to the evolution of (5.1.1):

Lemma 5.5 *Assume* $A_0 := A(0)$ *is an upper Hessenberg matrix and suppose it evolves according to* (5.1.1). *Then for each subdiagonal element* $A_{j+1,j}$ $j = 1, \ldots, n-1$ *the following happens: if* $(A_0)_{j+1,j} = 0$ *then* $(A(t))_{j+1,j} = 0$ *for all future times and if* $(A_0)_{j+1,j} \neq 0$, *then* $(A(t))_{j+1,j} \neq 0$ *for all future* finite *times.*

Proof: Using (5.1.1), after a straightforward computation we get

$$\frac{dA_{j+1,j}}{dt} = A_{j+1,j}\left[\left([A^\dagger, A]_{du}\right)_{j+1,j+1} - \left([A^\dagger, A]_{du}\right)_{jj}\right], \quad j = 1, \ldots, n-1. \tag{5.1.6}$$

From this, the lemma follows immediately: if $(A_0)_{j+1,j} = 0$, then $A_{j+1,j}$ stays zero. The second part follows from the existence and uniqueness for systems of ODEs of this form. ■

In the previous lemma, we have emphasized that if $(A_0)_{j+1,j} \neq 0$, then $(A(t))_{j+1,j} \neq 0$ for all future *finite* times, because in general it is not possible to predict the behavior of $\lim_{t\to+\infty}(A(t))_{j+1,j}$ simply from the form of the equation. However, in the next section we are going to prove that the ω-limit set is actually a singleton and that if $(A_0)_{j+1,j} \neq 0$, then $\lim_{t\to+\infty}(A(t))_{j+1,j}$ exists and it is different from zero.

5.2 The ω-limit Set is a Singleton

We study the convergence properties of the flow (5.1.1) starting with an initial datum A_0 which is an upper Hessenberg matrix with simple spectrum Λ and nonzero subdiagonal elements.

By Lemma 5.3 we know that if the flow (5.1.1) is initialized with A_0 with spectrum Λ, then the ω-limit set $\Omega(A_0)$ is contained in the compact manifold of normal matrices isospectral to A_0, i.e., $\Omega(A_0) \subset \mathcal{N}_\Lambda$.

First of all, we can give a more precise description of the ω-limit set after the following lemma:

Lemma 5.6 *Let A_0 be an upper Hessenberg matrix with simple spectrum Λ and nonzero subdiagonal elements. Then there is no T upper triangular and with determinant one and belonging to $\mathcal{G}_0$ such that $(TA_0T^{-1})_{j+1,j} = 0$, for $j = 1, \ldots, n-1$.*

Proof: A straightforward computation shows that

$$(TA_0T^{-1})_{j+1,j} = T_{j+1,j+1}(A_0)_{j+1,j}(T^{-1})_{jj} \tag{5.2.1}$$

for $j = 1, \ldots, n-1$. Since $(A_0)_{j+1,j} \neq 0$ and T has determinant one, all its diagonal entries are different from zero and the same is true for T^{-1}. In this way the claim is proved. ■

If $A(t)$ evolves according to (5.1.1), then $A(t) = T(t)A_0T^{-1}(t)$ for $T(t) \in \mathcal{G}_0 \subset GL_n(\mathbb{C})$. In general the ODE (5.1.1) does not lift to a unique equation for the evolution of $T(t)$, but different lifts provide the same evolution for $A(t)$. To see this, since $TT^{-1} = \mathbb{I}$, then $\dot{T}T^{-1} + T\dot{T}^{-1} = \mathbb{O}$ implies $\dot{T}T^{-1} = -T\dot{T}^{-1}$. So, $\dot{T}^{-1} = -T^{-1}\dot{T}T^{-1}$.

If $A = TA_0T^{-1}$ evolves following (5.1.1) and $\dot{A} = \dot{T}A_0T^{-1} + TA_0\dot{T}^{-1}$ then,

$$\begin{aligned} & [[A^\dagger, A]_{du}, A] = \dot{T}A_0T^{-1} + TA_0\dot{T}^{-1} \\ \implies & [[A^\dagger, A]_{du}, TA_0T^{-1}] = \dot{T}A_0T^{-1} + TA_0\dot{T}^{-1} \end{aligned}$$

Using $\dot{T}^{-1} = -T^{-1}\dot{T}T^{-1}$, we get

$$\begin{aligned} & [A^\dagger, A]_{du}TA_0T^{-1} - TA_0T^{-1}[A^\dagger, A]_{du} = \dot{T}A_0T^{-1} - TA_0T^{-1}\dot{T}T^{-1} \\ \implies & [A^\dagger, A]_{du}TA_0 - TA_0T^{-1}[A^\dagger, A]_{du}T = \dot{T}A_0 - TA_0T^{-1}\dot{T} \end{aligned}$$

Multiply both sides by T^{-1}, we get

$$
\begin{aligned}
&T^{-1}[A^\dagger, A]_{du} T A_0 - A_0 T^{-1}[A^\dagger, A]_{du} T = T^{-1}\dot{T} A_0 - A_0 T^{-1}\dot{T} \\
\implies &[T^{-1}[A^\dagger, A]_{du} T, A_0] = [T^{-1}\dot{T}, A_0] \\
\implies &[T^{-1}[A^\dagger, A]_{du} T, A_0] - [T^{-1}\dot{T}, A_0] = \mathbb{O} \\
\implies &[T^{-1}[A^\dagger, A]_{du} T - T^{-1}\dot{T}, A_0] = \mathbb{O}.
\end{aligned}
$$

Now, since we can take A_0 arbitrary claim that $T^{-1}[A^\dagger, A]_{du} T - T^{-1}\dot{T} = \mathbb{O}$ upto a scalar matrix, i.e., $T^{-1}[A^\dagger, A]_{du} T - T^{-1}\dot{T} = \gamma(t)\mathbb{I}$.

Let $A(t) = T(t) A_0 T(t)^{-1}$ and $A(t) = G(t) A_0 G(t)^{-1}$ for some $T, G \in \mathcal{G}_0 \subset GL_n(\mathbb{C})$. Then by uniqueness of the flow of (5.1.1) we have

$$
\begin{aligned}
&G(t) A_0 G(t)^{-1} = T(t) A_0 T(t)^{-1} \\
\implies &T(t)^{-1} G(t) A_0 G(t)^{-1} T(t) = A_0 \\
\implies &T(t)^{-1} G(t) A_0 \left(T(t)^{-1} G(t)\right)^{-1} = A_0.
\end{aligned}
$$

This shows that $T(t)^{-1}G(t) \in \mathrm{Stab}(A_0)$ for some $T, G \in \mathcal{G}_0 \subset GL_n(\mathbb{C})$.

Here the stabilizer is taken under the action conjugation which is also known as centralizer or normalizer. The stabilizer of an element a of group G under the action conjugation is defined by $\mathrm{Stab}_G(a) = \{g \in G | gag^{-1} = a\}$.

We impose this is true for all $A_0 \in GL_n(\mathbb{C})$, not just for the initial conditions we are considering. This implies

$$
T^{-1}G \in \bigcap_{A_0 \in GL_n(\mathbb{C})} \mathrm{Stab}_{\mathcal{G}_0}(A_0) \subset \bigcap_{A_0 \in \mathcal{G}_0} \mathrm{Stab}_{\mathcal{G}_0}(A_0) = Z(\mathcal{G}_0)
$$

where $Z(\mathcal{G}_0)$ is center of $\mathcal{G}_0$ (for the last equality see [19], pg 189). Now the center of the group of invertible upper triangular matrices is of the form $\lambda(t)\mathbb{I}$ for some $\lambda(t) \neq 0$ for all t. However, in our case T, G are in $\mathcal{G}_0$, so they have determinant one which implies that $\lambda(t) = \pm 1$ for all t, depending on the parity of n. Again $Z(GL_n(\mathbb{C})) = \lambda(t)\mathbb{I}$ for some $\lambda(t) \neq 0$ for all t, (for instance see [6], pg 73). Even if we do not use this information we have $T(t)^{-1}G(t) = \lambda(t)\mathbb{I}$, for some $\lambda(t) \neq 0$ for all t. This implies $G(t) = \lambda(t)T(t)$.

Denote $T(t) = T_1$ and $G(t) = T_2$.

Let $\dot{T}_1 = [A^\dagger, A]_{du}T_1$ and $\dot{T}_2 = \left([A^\dagger, A]_{du} + \gamma(t)\mathbb{I}\right) T_2$ for some continuous $\gamma(t)$. Then $T_2(t) = e^{\int_0^t \gamma(s)ds}T_1(t)$. (This can be obtained by integrating both differential equations for $\dot{T}_1$ and $\dot{T}_2$ and using same initial conditions $T_1(0) = T_2(0)$.)

Now, to check T_2 is a solution of $\dot{T}_2 = \left([A^\dagger, A]_{du} + \gamma(t)\mathbb{I}\right) T_2$ we differentiate $T_2(t) = e^{\int_0^t \gamma(s)ds}T_1(t)$ with respect to t.

$$
\begin{aligned}
\dot{T}_2(t) &= e^{\int_0^t \gamma(s)ds}\dot{T}_1(t) + \gamma(t)e^{\int_0^t \gamma(s)ds}T_1(t) \\
&= e^{\int_0^t \gamma(s)ds}[A^\dagger, A]_{du}T_1(t) + \gamma(t)e^{\int_0^t \gamma(s)ds}T_1(t) \\
&= e^{\int_0^t \gamma(s)ds}\left([A^\dagger, A]_{du} + \gamma(t)\mathbb{I}\right) T_1(t) \\
&= \left([A^\dagger, A]_{du} + \gamma(t)\mathbb{I}\right) e^{\int_0^t \gamma(s)ds}T_1(t) \\
&= \left([A^\dagger, A]_{du} + \gamma(t)\mathbb{I}\right) T_2(t).
\end{aligned}
$$

We are looking for $T(t)$ such that $A(t) = T(t)A_0T(t)^{-1}$.

Though $\gamma(t)$ is time dependent complex function but when we use this, it does not affect the form $A(t) = T(t)A_0T(t)^{-1}$. To see this, if we use $T_2(t) = \lambda(t)T_1(t)$

$$
\begin{aligned}
A(t) &= T_2(t)A_0T_2(t)^{-1} \\
&= \lambda(t)T_1(t)A_0\left(\lambda(t)T_1(t)\right)^{-1} \\
&= \lambda(t)T_1(t)A_0\lambda(t)^{-1}T_1(t)^{-1} \\
&= \lambda(t)\lambda(t)^{-1}T_1(t)A_0T_1(t)^{-1} \\
&= T_1(t)A_0T_1(t)^{-1}.
\end{aligned}
$$

Hence, $A(t) = T_1(t)A_0T_1(t)^{-1} \iff A(t) = T_2(t)A_0T_2(t)^{-1}$. This proves our claim.

Therefore, we can use as an ODE for the time evolution of $T(t)$ the following one

$$\frac{dT}{dt} = [A^\dagger, A]_{du} T = \left[(TA_0T^{-1})^\dagger, TA_0T^{-1}\right]_{du} T, \tag{5.2.2}$$

where (5.2.2) holds on $[0, t_{\max})$, the maximal interval of existence and where $T(0)$ is the identity, to be consistent with $A(t) = T(t)A_0T^{-1}(t)$. The fact that $A(t)$ is bounded for all future times, does not imply that $T(t)$ and $T(t)^{-1}$ are also bounded for all future times.

In the following we outline and prove that $T(t)$ and $T(t)^{-1}$, with $A(t) := T(t)A_0T(t)^{-1}$ remain bounded for $t \geq 0$ based on the fact that $\|[A^\dagger(t), A(t)]_{du}\|_F$ converges to zero exponentially fast. Indeed, considering $\|T\|_F^2 := \operatorname{trace}(TT^\dagger)$ we have

$$\frac{d\|T\|_F^2}{dt} = 2\,\mathrm{Re}\left(\operatorname{trace}([A^\dagger, A]_{du}TT^\dagger)\right) = 2\,\mathrm{Re}\langle [A^\dagger, A]_{du}, (TT^\dagger)^\dagger\rangle = 2\,\langle [A^\dagger, A]_{du}, TT^\dagger\rangle$$

$$\leq 2\,\|TT^\dagger\|_F\ \|[A^\dagger, A]_{du}\|_F \leq 2\,\|T\|_F^2\ \|[A^\dagger, A]_{du}\|_F,$$

using the Cauchy-Schwartz inequality for the Hermitian scalar product $\langle A, B\rangle :=$ $\operatorname{trace}(AB^\dagger)$ and the sub-multiplicative property of the Frobenius norm: $\|AB\|_F \leq$ $\|A\|_F\|B\|_F$. After a straightforward simplification, we get immediately

$$\frac{1}{\|T(t)\|_F}\frac{d\|T(t)\|_F}{dt} \leq \|[A^\dagger(t), A(t)]_{du}\|_F \tag{5.2.3}$$

and integrating both sides of (5.2.3) along the solution of (5.1.1) starting at an initial condition A_0 we obtain the following estimate:

$$\ln(\|T(t)\|_F) - \ln(\|T(0)\|_F) \leq \int_0^t \|[A^\dagger(s), A(s)]_{du}\|_F\ ds. \tag{5.2.4}$$

Therefore, to prove that $\|T(t)\|_F$ remains bounded for future time, it is sufficient to show that the integral on the right hand side of (5.2.4) is convergent as t goes to

$+\infty$. To prove that $\|T^{-1}(t)\|_F$ remains bounded, just observe that from $TT^{-1} = \mathbb{I}$ one obtains $\frac{dT^{-1}}{dt} = -T^{-1}[A^\dagger, A]_{du}$. Using the same estimates above we get

$$\frac{d\|T^{-1}\|_F^2}{dt} \leq 2\,\|T^{-1}\|_F^2\;\|[A^\dagger, A]_{du}\|_F,$$

from which the counterpart of inequality (5.2.4) for T^{-1} follows:

$$\ln(\|T(t)^{-1}\|_F) - \ln(\|T(0)^{-1}\|_F) \leq \int_0^t \|[A^\dagger(s), A(s)]_{du}\|_F\, ds. \tag{5.2.5}$$

Next, we are going to prove that $\|[A^\dagger(t), A(t)]_{du}\|_F$ converges to zero exponentially fast along a solution of (5.1.1) starting from an initial condition A_0 with simple spectrum.

The main idea is to prove that the compact manifold $\mathcal{N}_\Lambda$ of all normal matrices isospectral to A_0 is exponentially attracting for the flow (5.1.1), where A_0 is any upper Hessenberg matrix with simple spectrum. This manifold contains all the ω-limit sets for initial data A_0 with simple spectrum due to Lemma 5.3.

Now we recall some results and definitions from [13].

Definition 5.7 *[13] Let $\mathcal{Q}$ be an embedded submanifold inside the phase space $\mathcal{M}$ of a vector field $\frac{dx}{dt} = F(x)$. Suppose that $\mathcal{Q}$ is invariant for the local flow ϕ_F^t of $\frac{dx}{dt} = F(x)$, then $\mathcal{Q}$ is called exponentially attracting for the vector field F if there exists a neighborhood W of $\mathcal{Q}$ and positive constants K and γ such that for any point $x_0 = x(0) \in W$ and for $t \geq 0$ the following inequality holds:*

$$d(\phi_F^t(x_0), \mathcal{Q}) \leq K e^{-\gamma t} d(x_0, \mathcal{Q}),$$

where d is a distance function.

Following [13], consider along with the vector field $\frac{dx}{dt} = F(x)$ the system

$$\frac{dx}{dt} = F(x), \quad \frac{d\xi}{dt} = DF(x)\xi \tag{5.2.6}$$

where DF is the Jacobian of F. System (5.2.6) can be interpreted as the family of all linearizations of $\frac{dx}{dt} = F(x)$ along its solutions; in fact if $x(t)$ is a solution of $\frac{dx}{dt} = F(x)$, then $\frac{d\xi}{dt} = DF(x(t))\xi$ is a nonautonomous linear system. In the following, $x(t)$ will be assumed to be a solution lying on $\mathcal{Q}$. For each $x \in \mathcal{Q}$, let us denote with $T_x\mathcal{Q}$ the tangent space at x to $\mathcal{Q}$ and $N_x\mathcal{Q}$ its orthogonal complement (for a choice of a metric in the environment $\mathcal{M}$, which in our case is given by the Hermitian form trace$(AB^\dagger)$ on the space of all complex matrices). Let us also denote with $P_x : T_x\mathcal{M} \to N_x\mathcal{Q}$ the projection operator sending vector $\xi \in T_x\mathcal{M}$ onto $N_x\mathcal{Q}$.

Definition 5.8 *[13] An invariant manifold $\mathcal{Q}$ of $\frac{dx}{dt} = F(x)$ is called exponentially stable in linear approximation if for any trajectory $x(t)$ lying on $\mathcal{Q}$ and any $\xi(0)$, the corresponding solution of system* (5.2.6) *satisfies for $t \geq 0$ the following inequality*

$$\|\nu(t)\| \leq Ke^{-\beta t}\|\nu(0)\|, \quad \nu(t) = P_{x(t)}\xi(t), \quad \beta > 0 \tag{5.2.7}$$

where the constants K and β can be chosen to be independent on the choice of $x(0)$ and $\xi(0)$. The norm is the one induced by the metric restricted to the normal bundle.

The following result highlights the connection between the two definitions above:

Theorem 5.9 *[13] An invariant compact manifold $\mathcal{Q}$ is exponentially attracting if and only if $\mathcal{Q}$ is exponentially stable in linear approximation.*

We apply the previous result taking $\mathcal{N}_\Lambda$ as the manifold $\mathcal{Q}$. In this case $\mathcal{Q}$ consists of a manifold of equilibria, so to check that property (5.2.7) holds it is sufficient to linearize the flow (5.1.1) at an arbitrary equilibrium point $N_0 \in \mathcal{N}_\Lambda$ and check that the normal directions to $T_{N_0}\mathcal{N}_\Lambda = \{[W, N_0] : W^\dagger = -W\}$ are decreasing exponentially fast (see for instance [18] for a detailed description on how to compute these tangent spaces).

Lemma 5.10 *Call N_0 any point in $\mathcal{N}_\Lambda$ and denote with $T_{N_0}\mathcal{N}_\Lambda$ the tangent space to $\mathcal{N}_\Lambda$ at N_0. Consider any vector V tangent to the space $M_n(\mathbb{C})$ at N_0, $V = [U, N_0]$. Then its projection to the orthogonal complement $N_{N_0}\mathcal{N}_\Lambda$ of $T_{N_0}\mathcal{N}_\Lambda$ in the space of all matrices is given by*

$$\pi_N(V) := \left[\frac{U+U^\dagger}{2}, N_0\right] - \frac{\langle[\frac{U+U^\dagger}{2}, N_0], [\frac{U-U^\dagger}{2}, N_0]\rangle}{\langle[\frac{U-U^\dagger}{2}, N_0], [\frac{U-U^\dagger}{2}, N_0]\rangle}\left(\left[\frac{U-U^\dagger}{2}, N_0\right]\right),$$

where the orthogonal complement $N_{N_0}\mathcal{N}_\Lambda$ of $T_{N_0}\mathcal{N}_\Lambda$ is taken with respect to the Hermitian metric $\langle A, B\rangle := \mathrm{trace}(AB^\dagger)$.

Proof: Decompose V as $V = \left[\frac{U-U^\dagger}{2}, N_0\right] + \left[\frac{U+U^\dagger}{2}, N_0\right]$ and use the following notations to simplify writing, $V_T := \left[\frac{U-U^\dagger}{2}, N_0\right]$, $\check{V}_T := \left[\frac{U+U^\dagger}{2}, N_0\right]$, $V_N := \pi_N(V)$. Using the above notations, we have to show $V_N = \check{V}_T - \frac{\langle\check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle}V_T$. Note that we are using the Hermitian metric $\langle A, B\rangle := \mathrm{trace}(AB^\dagger)$. This metric has the property $\langle\lambda A, B\rangle = \lambda\langle A, B\rangle$, $\langle A, \lambda B\rangle = \bar{\lambda}\langle A, B\rangle$ and $\langle A, B\rangle = \overline{\langle B, A\rangle}$.

The space $T_{N_0}\mathcal{N}_\Lambda$ is described by vectors of the form $[W, N_0]$ where W varies in the Lie algebra of skew-Hermitian matrices. Therefore, the projection $\pi_T : T_{N_0}M_n(\mathbb{C}) \to T_{N_0}\mathcal{N}_\Lambda$ is given by

$$\pi_T(V) = \pi_T([U, N_0]) = \left[\frac{U-U^\dagger}{2}, N_0\right] \quad \text{with } V = [U, N_0] \in T_{N_0}M_n(\mathbb{C}),$$

while the orthogonal complement $\pi_N(V)$ of $\pi_T(V)$ is given by Gram-Schmidt process

$$V_N = \pi_N(V) = V - \frac{\langle V, V_T\rangle}{\langle V_T, V_T\rangle} V_T = V_T + \check{V}_T - \frac{\langle V_T + \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T =$$
$$= V_T + \check{V}_T - \frac{\langle V_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T - \frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T = \check{V}_T - \frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T.$$

Therefore, $V_N = \check{V}_T - \frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T$.

We do not need to check that V_N and V_T are orthogonal to each other because V_N is obtained by Gram-Schmidt process. ∎

To prove that $\mathcal{N}_\Lambda$ is indeed linearly exponentially stable, we need to linearize the flow (5.1.1) at a point $N_0 \in \mathcal{N}_\Lambda$. Recall that such an N_0 is a normal matrix with simple spectrum. In order to obtain the linearization, we write a first order deformation of N_0 as $dN_0 = N_0 + [U, N_0]$ and we obtain an equation for $[U, N_0]$:

Lemma 5.11 *The linearization of* (5.1.1) *at* N_0 *is given by*

$$\frac{d}{dt}[U(t), N_0] = -\left[\left[[U + U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right]. \tag{5.2.8}$$

Proof: Equation (5.2.8) can be obtained substituting $N_0 + [U, N_0]$ instead of A in (5.1.1).

$$\begin{aligned}
\frac{d}{dt}\left(N_0 + [U(t), N_0]\right) &= \left[\left([(N_0 + [U, N_0])^\dagger, N_0 + [U, N_0]]\right)_{du}, N_0 + [U, N_0]\right] \\
&= \left[\left([N_0^\dagger + [U, N_0]^\dagger, N_0 + [U, N_0]]\right)_{du}, N_0 + [U, N_0]\right] \\
&= \left[\left([N_0^\dagger - [U^\dagger, N_0^\dagger], N_0 + [U, N_0]]\right)_{du}, N_0 + [U, N_0]\right].
\end{aligned}$$

Now use the fact that N_0 is an equilibrium so that $\frac{dN_0}{dt} = 0$ and normality of N_0 to get:

$$
\begin{aligned}
&\frac{d}{dt}[U(t), N_0] \\
&= \left[\left([N_0^\dagger, N_0] + \left[N_0^\dagger, [U, N_0]\right] - \left[[U^\dagger, N_0^\dagger], N_0\right] - \left[[U^\dagger, N_0^\dagger], [U, N_0]\right]\right)_{du}, N_0 + [U, N_0]\right] \\
&= \left[\left(\left[N_0^\dagger, [U, N_0]\right] - \left[[U^\dagger, N_0^\dagger], N_0\right] - \left[[U^\dagger, N_0^\dagger], [U, N_0]\right]\right)_{du}, N_0 + [U, N_0]\right] \\
&= \left[\left[N_0^\dagger, [U, N_0]\right]_{du} - \left[[U^\dagger, N_0^\dagger], N_0\right]_{du} - \left[[U^\dagger, N_0^\dagger], [U, N_0]\right]_{du}, N_0 + [U, N_0]\right] \\
&= \left[\left[N_0^\dagger, [U, N_0]\right]_{du}, N_0\right] - \left[\left[[U^\dagger, N_0^\dagger], N_0\right]_{du}, N_0\right] - \left[\left[[U^\dagger, N_0^\dagger], [U, N_0]\right]_{du}, N_0\right] \\
&+ \left[\left[N_0^\dagger, [U, N_0]\right]_{du}, [U, N_0]\right] - \left[\left[[U^\dagger, N_0^\dagger], N_0\right]_{du}, [U, N_0]\right] \\
&- \left[\left[[U^\dagger, N_0^\dagger], [U, N_0]\right]_{du}, [U, N_0]\right].
\end{aligned}
$$

Now, collecting the terms linear in $[U, N_0]$ and disregarding those that have a quadratic or cubic dependence on $[U, N_0]$, we get:

$$
\begin{aligned}
\frac{d}{dt}[U(t), N_0] &= \left[\left[N_0^\dagger, [U, N_0]\right]_{du}, N_0\right] - \left[\left[[U^\dagger, N_0^\dagger], N_0\right]_{du}, N_0\right] \\
&= -\left[\left[[U, N_0], N_0^\dagger\right]_{du}, N_0\right] - \left[\left[[U^\dagger, N_0^\dagger], N_0\right]_{du}, N_0\right].
\end{aligned}
$$

Using the Jacobi identity: $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ with $A = U^\dagger$, $B = N_0^\dagger$ and $C = N_0$ and $[N_0^\dagger, N_0] = 0$, we get $\left[[U^\dagger, N_0^\dagger], N_0\right] = \left[[U^\dagger, N_0], N_0^\dagger\right]$. Applying this identity to the above computation and simplifying we obtain

$$
\begin{aligned}
\frac{d}{dt}[U(t), N_0] &= -\left[\left[[U, N_0], N_0^\dagger\right]_{du}, N_0\right] - \left[\left[[U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right] \\
&= -\left[\left(\left[[U, N_0], N_0^\dagger\right] + \left[[U^\dagger, N_0], N_0^\dagger\right]\right)_{du}, N_0\right] \\
&= -\left[\left(\left[[U, N_0] + [U^\dagger, N_0], N_0^\dagger\right]\right)_{du}, N_0\right] \\
&= -\left[\left[[U + U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right].
\end{aligned}
$$

This proves the lemma. Moreover, using the same computation or replacing U with

$U^\dagger$ we have

$$\frac{d}{dt}[U(t)^\dagger, N_0] = -\left[\left[[U+U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right].$$

■

We want to show that if $[U, N_0]$ evolves according to the linearization (5.2.8) then its normal component

$$V_N(t) := \pi_N([U, N_0]) = \check{V}_T - \frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T \in N_{N_0}\mathcal{N}_\Lambda \tag{5.2.9}$$

is converging to zero exponentially fast. This will allow us to conclude that $\mathcal{N}_\Lambda$ is linearly exponentially stable.

We have the following key result:

Proposition 5.12 *If* $[U, N_0]$ *evolves according to* (5.2.8)*, then the normal component* $V_N(t)$ *converges to zero exponentially fast.*

Proof: First of all, observe the following: We have, $V_N = \check{V}_T - \frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T$. Use triangle inequality and Cauchy-Schwartz inequality to get

$$\|V_N\| \leq \|\check{V}_T\| + \|\frac{\langle \check{V}_T, V_T\rangle}{\langle V_T, V_T\rangle} V_T\| = \|\check{V}_T\| + \frac{|\langle \check{V}_T, V_T\rangle|}{\|V_T\|^2}\|V_T\| \leq \|\check{V}_T\| + \frac{\|\check{V}_T\|\|V_T\|}{\|V_T\|^2}\|V_T\|.$$

This gives

$$\|V_N\| \leq 2\|\check{V}_T\| \qquad \Longrightarrow \|V_N\|_F^2 \leq 4\|\check{V}_T\|_F^2.$$

To show that the quadratic form $\|V_N\|_F^2$ is exponentially decreasing, we show that the quadratic form $\|\check{V}_T\|_F^2$ is exponentially decreasing. For that, we will show $\frac{d\|\check{V}_T\|_F^2}{dt}$ is a negative definite quadratic form.

First we claim that if $[U, N_0]$ evolves according to (5.2.8), then $\check{V}_T(t)$ evolve according to

$$\frac{d\check{V}_T}{dt} = -2\left[[\check{V}_T, N_0^\dagger]_{du}, N_0\right].$$

To prove the claim, using $\check{V}_T(t) = \left[\frac{U(t)+U(t)^\dagger}{2}, N_0\right] = \frac{1}{2}[U(t), N_0] + \frac{1}{2}[U(t)^\dagger, N_0]$ and the expression for $\frac{d}{dt}[U(t), N_0]$ together with the one for $\frac{d}{dt}\left([U(t)^\dagger, N_0]\right)$ obtained in Lemma 5.11, we get:

$$\begin{aligned}
\frac{d\check{V}_T}{dt} &= \frac{1}{2}\frac{d}{dt}[U(t), N_0] + \frac{1}{2}\frac{d}{dt}[U(t)^\dagger, N_0] \\
&= -\frac{1}{2}\left[\left[[U+U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right] - \frac{1}{2}\left[\left[[U+U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right] \\
&= -\left[\left[[U+U^\dagger, N_0], N_0^\dagger\right]_{du}, N_0\right] = -2\left[\left[\left[\frac{U+U^\dagger}{2}, N_0\right], N_0^\dagger\right]_{du}, N_0\right] \\
&= -2\left[[\check{V}_T, N_0^\dagger]_{du}, N_0\right].
\end{aligned}$$

To prove the proposition, observe that the Lie derivative of the Frobenius norm $\|\check{V}_T(t)\|_F^2 = \text{trace}(\check{V}_T\check{V}_T^\dagger)$ along the vector field (5.2.8) is

$$\frac{d\|\check{V}_T\|_F^2}{dt} = 2\,\text{Re}\,\text{trace}\left(\dot{\check{V}}_T\check{V}_T^\dagger\right).$$

We obtain

$$\begin{aligned}
\frac{d\|\check{V}_T\|_F^2}{dt} &= 2\,\text{Re}\,\text{trace}\left(\left(-2\left[[\check{V}_T, N_0^\dagger]_{du}, N_0\right]\right)\check{V}_T^\dagger\right) \\
&= -4\,\text{Re}\,\text{trace}\left(\left[[\check{V}_T, N_0^\dagger]_{du}, N_0\right]\check{V}_T^\dagger\right) \\
&= 4\,\text{Re}\,\text{trace}\left(\left[N_0, [\check{V}_T, N_0^\dagger]_{du}\right]\check{V}_T^\dagger\right).
\end{aligned}$$

Using $\text{trace}([A,B]C) = \text{trace}([C,A]B)$, we get

$$= 4\,\text{Re}\,\text{trace}\left([\check{V}_T^\dagger, N_0][\check{V}_T, N_0^\dagger]_{du}\right)$$

$$= -4\,\text{Re}\,\text{trace}\left([\check{V}_T, N_0^\dagger]^\dagger[\check{V}_T, N_0^\dagger]_{du}\right).$$

Note that $[\check{V}_T, N_0^\dagger]$ is Hermitian, indeed we have

$$[\check{V}_T, N_0^\dagger]^\dagger = -[\check{V}_T^\dagger, N_0] = -\left[\left[\frac{U+U^\dagger}{2}, N_0\right]^\dagger, N_0\right] = \left[\left[\frac{U+U^\dagger}{2}, N_0^\dagger\right], N_0\right].$$

Using the Jacobi identity to get $\left[\left[\frac{U+U^\dagger}{2}, N_0^\dagger\right], N_0\right] = \left[\left[\frac{U+U^\dagger}{2}, N_0\right], N_0^\dagger\right]$, we obtain

$$[\check{V}_T, N_0^\dagger]^\dagger = \left[\left[\frac{U+U^\dagger}{2}, N_0\right], N_0^\dagger\right] = [\check{V}_T, N_0^\dagger].$$

Hence,

$$\frac{d\|\check{V}_T\|_F^2}{dt} = -4\,\mathrm{Re}\,\mathrm{trace}\left([\check{V}_T, N_0^\dagger]^\dagger[\check{V}_T, N_0^\dagger]_{du}\right) = -4\,\mathrm{Re}\,\mathrm{trace}\left([\check{V}_T, N_0^\dagger][\check{V}_T, N_0^\dagger]_{du}\right).$$

Using the same argument about a Hermitian matrix X used in the proof of Lemma 5.3, we obtain

$$\frac{d\|\check{V}_T\|_F^2}{dt} = -4\,\mathrm{Re}\,\mathrm{trace}\left([\check{V}_T, N_0^\dagger]_{du}[\check{V}_T, N_0^\dagger]_{du}^\dagger\right) = -4\|[\check{V}_T, N_0^\dagger]_{du}\|_F^2 \leq 0.$$

This proves the quadratic form is negative semidefinite. To prove that the quadratic form $\frac{d\|\check{V}_T(t)\|_F^2}{dt}$ is indeed negative definite on $N_{N_0}\mathcal{N}_\Lambda$ we observe the following. If $\frac{d\|\check{V}_T(t)\|_F^2}{dt} = 0$ then $\|[\check{V}_T, N_0^\dagger]_{du}\|_F^2 = 0$ which implies $[\check{V}_T, N_0^\dagger] = 0$ since $[\check{V}_T, N_0^\dagger]$ is Hermitian. We want to show that $[\check{V}_T, N_0^\dagger] = 0$ implies $\check{V}_T = 0$. This gives $V_N = 0$, which shows that the quadratic form is indeed negative definite.

Since N_0 is normal, there exists a unitary matrix W such that $W^\dagger N_0 W = \Lambda$ where Λ is a diagonal matrix. Therefore, $\left(W^\dagger N_0 W\right)^\dagger = \Lambda^\dagger$ implies $W^\dagger N_0^\dagger W = \bar{\Lambda}$. Since N_0 is isospectral to A_0, N_0 has simple spectrum by hypothesis and the diagonal elements of Λ are the eigenvalue of A_0. From $[\check{V}_T, N_0^\dagger] = 0$ we get

$$\begin{aligned}
&\check{V}_T N_0^\dagger - N_0^\dagger \check{V}_T = 0, \\
\Longrightarrow\ & W^\dagger \check{V}_T W W^\dagger N_0^\dagger W - W^\dagger N_0^\dagger W W^\dagger \check{V}_T W = 0, \\
\Longrightarrow\ & C\bar{\Lambda} - \bar{\Lambda} C = 0,
\end{aligned}$$

where $C = W^\dagger \check{V}_T W$. Writing $C\bar{\Lambda} - \bar{\Lambda}C = 0$ explicitly we get $\sum_{k=1}^{n} \left(C_{jk}\bar{\Lambda}_{kl} - \bar{\Lambda}_{jk} C_{kl} \right) = 0$. Since $\bar{\Lambda}_{kl} = 0$ for all $k \neq l$ and $\bar{\Lambda}_{jk} = 0$ for all $k \neq j$, the sum $\sum_{k=1}^{n} \left(C_{jk}\bar{\Lambda}_{kl} - \bar{\Lambda}_{jk} C_{kl} \right) = 0$ reduces to

$$C_{jl}\left(\bar{\Lambda}_{ll} - \bar{\Lambda}_{jj}\right) = 0.$$

Because of simplicity of the spectrum, we have $C_{jl} = 0$ for all $l \neq j$. So, C is a diagonal matrix.

Moreover, we have $C = W^\dagger \check{V}_T W = W^\dagger \left[\frac{U+U^\dagger}{2}, N_0 \right] W = W^\dagger \left(\frac{U+U^\dagger}{2} N_0 - N_0 \frac{U+U^\dagger}{2} \right) W$. Thus:

$$\begin{aligned}
C &= W^\dagger \left(\frac{U+U^\dagger}{2} W W^\dagger N_0 - N_0 W W^\dagger \frac{U+U^\dagger}{2} \right) W \\
&= W^\dagger \frac{U+U^\dagger}{2} W W^\dagger N_0 W - W^\dagger N_0 W W^\dagger \frac{U+U^\dagger}{2} W \\
&= \left[W^\dagger \frac{U+U^\dagger}{2} W,\ \ W^\dagger N_0 W \right] = \left[W^\dagger \frac{U+U^\dagger}{2} W,\ \ \Lambda \right] = [H, \Lambda],
\end{aligned}$$

where $H = W^\dagger \frac{U+U^\dagger}{2} W$ is Hermitian. Therefore $C_{jl} = \sum_{k=1}^{n} \left(H_{jk}\Lambda_{kl} - \Lambda_{jk} H_{kl} \right)$, and since $\Lambda_{kl} = 0$ for all $k \neq l$ and $\Lambda_{jk} = 0$ for all $k \neq j$ we have $C_{jl} = H_{jl}\Lambda_{ll} - \Lambda_{jj} H_{jl}$. But C is diagonal therefore $C_{jl} = 0$ for all $j \neq l$ and this gives $H_{jl}\left(\Lambda_{ll} - \Lambda_{jj}\right) = 0$ for all $j \neq l$. Again, because of the simplicity of the spectrum we get $H_{jl} = 0$ for all $l \neq j$. So, H is a diagonal matrix. Since H and Λ are diagonal matrices and diagonal matrices commute, we have that $[H, \Lambda] = \mathbb{O}$.

This proves $C = [H, \Lambda] = \mathbb{O}$. If $C = \mathbb{O}$ then $\check{V}_T = WCW^\dagger = 0$. This means that the quadratic form $\frac{d\|\check{V}_T(t)\|_F^2}{dt}$ is indeed negative definite on $N_{N_0}\mathcal{N}_\Lambda$ and this shows that $\check{V}_T$ converges to zero exponentially fast due to Lyapunov linearization theorem. Inequality $\|V_N\| \leq 2\|\check{V}_T\|$ proves that V_N converges to zero exponentially fast. ■

We can now prove the following:

Theorem 5.13 *The manifold $\mathcal{N}_\Lambda$ is exponentially attracting for the flow* (5.1.1)

Proof: From Proposition 5.12 we have that $\check{V}_T(t)$ is converging to zero exponentially fast because the quadratic form $\frac{d\|\check{V}_T(t)\|_F^2}{dt}$ is negative definite on $N_{N_0}\mathcal{N}_\Lambda$, for each $N_0 \in \mathcal{N}_\Lambda$. In particular the eigenvalues of the quadratic form $\frac{d\|\check{V}_T(t)\|_F^2}{dt}$ are negative and they are bounded away globally from 0 as N_0 varies on $\mathcal{N}_\Lambda$ because $\mathcal{N}_\Lambda$ is compact and the quadratic form is continuous. Therefore there exist positive constants $\tilde{K}$ and $\tilde{\beta}$, independent on N_0 such that the bound in the Definition 5.8 holds (see for instance Observation 9.3 in [13]) for $\check{V}_T$. From the inequality $\|V_N\| \leq 2\|\check{V}_T\|$, there exist positive constants K and β, independent on N_0 such that the bound in the Definition 5.8 holds for V_N. Therefore $\mathcal{N}_\Lambda$ is linearly exponentially stable and by Theorem 5.9 it follows that $\mathcal{N}_\Lambda$ is exponentially attracting. ■

The time evolution of $\|\check{V}_T(t)\|_F$ is related to the time evolution of $\|[A(t)^\dagger, A(t)]_{du}\|_F$. Indeed we have the following lemma.

Lemma 5.14 *Let $A(t)$ be a solution of the flow* (5.1.1)*. The quantity $[A(t)^\dagger, A(t)]_{du}$ converges to zero exponentially fast if $\check{V}_T(t)$, evolving according to* (5.2.8) *converges to zero exponentially fast.*

Proof: First note that $\mathcal{N}_\Lambda$ is compact. So, $d(A, \mathcal{N}_\Lambda)$ is well defined, where d is the distance induced by the Hermitian metric. To prove $[A(t)^\dagger, A(t)]_{du}$ converges to zero exponentially fast, we will prove $\|[A(t)^\dagger, A(t)]\|_F$ is bounded by $d(A, \mathcal{N}_\Lambda)$ up to a constant.

Observe that $d(A, \mathcal{N}_\Lambda) = d(A, M)$ for some $M \in \mathcal{N}_\Lambda$. (Here we are using M instead of N_0 because there could be many such normal matrices M which give $d(A, \mathcal{N}_\Lambda) = d(A, M)$).

By a standard result in Differential Geometry, we have that $(A - M) \perp T_M\mathcal{N}_\Lambda$, i.e., a matrix M that realizes the distance of A from $\mathcal{N}_\Lambda$ has to satisfy the property that the vector $A - M$ is orthogonal to $T_M\mathcal{N}_\Lambda$, with respect to the real part of the Hermitian form. Write A as $A = M + (A - M)$, then

$$\begin{aligned}
[A^\dagger, A] &= [M^\dagger + (A - M)^\dagger, M + (A - M)] \\
&= [M^\dagger, (A - M)] + [(A - M)^\dagger, M] + [(A - M)^\dagger, (A - M)],
\end{aligned}$$

since M is normal. Using the fact that Frobenius norm is sub-multiplicative, that $\|A^\dagger\|_F = \|A\|_F$ and the triangle inequality we get the estimate:

$$\begin{aligned}
\|[A^\dagger, A]\|_F &\leq \|[M^\dagger, (A - M)]\|_F + \|[(A - M)^\dagger, M]\|_F + \|[(A - M)^\dagger, (A - M)]\|_F \\
&\leq 2\|M\|_F\|A - M\|_F + 2\|M\|_F\|A - M\|_F + 2\|A - M\|_F^2 \\
&= 4\|M\|_F\|A - M\|_F + 2\|A - M\|_F^2.
\end{aligned}$$

Since the function $\|\cdot\|_F : \mathcal{N}_\Lambda \to \mathbb{R}$ given by $M \mapsto \|M\|_F$ is continuous and $\mathcal{N}_\Lambda$ is compact, it has a maximum C on $\mathcal{N}_\Lambda$. Hence,

$$\|[A^\dagger, A]\|_F \leq 4Cd(A, \mathcal{N}_\Lambda) + 2d(A, \mathcal{N}_\Lambda)^2.$$

Using the fact that the manifold $\mathcal{N}_\Lambda$ is exponentially attracting for the flow (5.1.1), we can say that $d(A, \mathcal{N}_\Lambda)$ converges to zero exponentially fast, so is $\|[A^\dagger, A]\|_F$. This proves the lemma. ■

Let us remark that Lemma 5.14 proves that the integral $\int_0^t \|[A^\dagger(s), A(s)]_{du}\|_F \, ds$ is convergent for $t \to +\infty$ along a solution of (5.1.1), for any initial data A_0 with simple spectrum.

Combining the results so far obtained, we can prove the following

Theorem 5.15 *Let $A(t)$ be the solution of* (5.1.1) *starting from an initial condition A_0 which is upper Hessenberg, with simple spectrum and nonzero codiagonal elements. Then $\lim_{t\to+\infty} A(t)$ converges exponentially fast to the set of upper Hessenberg normal matrices isospectral to A_0 with nonzero lower codiagonal elements. Moreover, if A_0 has its simple spectrum contained in a line $l \subset \mathbb{C}$, then $\lim_{t\to+\infty} A(t)$ converges exponentially fast to the set of tridiagonal normal matrices isospectral to A_0 with nonzero codiagonal elements.*

Proof: By Lemma 5.14 we know that $\|[A(t)^\dagger, A(t)]_{du}\|_F$ is converging exponentially fast to zero. In particular, this implies that $\|T(t)\|_F$ and $\|T^{-1}(t)\|_F$ remain bounded because of inequality (5.2.4) and inequality (5.2.5), respectively. Therefore the eigenvalues of $T(t)$ and $T^{-1}(t)$ remain bounded and bounded away from zero. This allows us to apply Lemma 5.6 in the limit for $t \to +\infty$ and to conclude in particular $\lim_{t\to+\infty} A(t)_{j+1,j} \neq 0$ for $j = 1, \ldots, n-1$. Moreover, since by Lemma 5.3 $A(t)$ converges to the set of normal matrices isospectral to A_0 and the subdiagonal elements of $A(t)$ can not be zero by Lemma 5.5, we have that $A(t)$ has to converge to the set of normal matrices isospectral to A_0 with nonzero lower codiagonal elements. The second part of the claim in the theorem follows combining the previous lemma with Lemma 5.2 and Lemma 5.3. ∎

Our final goal is to show that the ω-limit set for an initial condition A_0 as above is indeed a single point. Since $\mathcal{N}_\Lambda$ is normally hyperbolic, because it is exponentially attracting, one could invoke the theory of normally hyperbolic manifolds to claim that the ω-limit set is in this case a singleton. The following theorem gives an elementary proof.

Theorem 5.16 *Let A_0 be an upper Hessenberg matrix with simple complex spectrum. Then $\Omega(A_0)$ is a singleton and it is a normal matrix, isospectral to A_0 with nonzero lower codiagonal elements corresponding to the nonzero subdiagonal elements of A_0. Furthermore if A_0 has spectrum contained in a line $l \subset \mathbb{C}$, then $\Omega(A_0)$ is a tridiagonal normal matrix.*

Proof: The only claim we need to prove is that $\Omega(A_0)$ is a singleton. Since the Frobenius norm is sub-multiplicative we obtain

$$\|[[A^\dagger, A]_{du}, A]\|_F \leq 2\|[A^\dagger, A]_{du}\|_F \ \|A\|_F.$$

Therefore we have:

$$\lim_{t\to+\infty} \int_0^t \|[[A^\dagger(s), A(s)]_{du}, A(s)]\|_F \ ds \leq 2 \lim_{t\to+\infty} \int_0^t \|[A^\dagger(s), A(s)]_{du}\|_F \|A(s)\|_F \ ds \leq$$

$$\leq \lim_{t\to+\infty} \int_0^t \|[A^\dagger(s), A(s)]_{du}\|_F K \ ds < +\infty,$$

since $\|A\|_F$ can be bounded by a constant K due to Lemma 5.3, and the integral of $\|[A^\dagger, A]_{du}\|_F$ along a solution is convergent by Lemma 5.14. This is enough to conclude that $\Omega(A_0)$ is indeed a singleton, since the convergence of the improper integral above says that the length of the solution curve is finite. ■

Therefore the flow (5.1.1) performs an explicit deformation from an upper Hessenberg matrix to a normal upper Hessenberg matrix preserving the spectrum.

Remark 5.17 *An immediate consequence of Theorem 5.16 is that if A_0 is a real upper Hessenberg matrix but with complex spectrum, then the $\Omega(A_0)$ is a singleton and it is a normal matrix, isospectral to A_0 with nonzero lower codiagonal elements corresponding to the nonzero subdiagonal elements of A_0. In general $\Omega(A_0)$ is not a tridiagonal matrix, unless the spectrum of A_0 is real (this last case was analyzed in* *detail in [1]), or unless the spectrum of* A_0 *is imaginary. Furthermore if the spectrum of* A_0 *is imaginary, then the singleton* $\Omega(A_0)$ *is given by a skew-symmetric tridiagonal matrix. Indeed, since a normal matrix with imaginary spectrum is necessarily skew-Hermitian, and since* A_0 *is assumed to be real and the flow* (5.1.1) *preserves real matrices, we have that* $\Omega(A_0)$ *has to be skew-Hermitian tridiagonal and real and therefore as claimed.*

The following proposition answers a question posed in [1] about the construction of *even dimensional real skew-symmetric tridiagonal matrices* with simple imaginary spectrum and given signs pattern for the codiagonal elements. Essentially it was observed numerically that the real analogue of (5.1.1) can be used to construct this kind of matrices in the following way. For simplicity we consider just the case of 6×6 matrix to show how the procedure works. Suppose one wants to construct such a matrix with spectrum $\pm i\lambda_1, \pm i\lambda_2, \pm i\lambda_3$ and with lower codiagonal signs pattern given by $+, -, +, +, +$. Then one constructs a tridiagonal matrix A_0 with diagonal given by all zeroes, upper codiagonal given by $-\lambda_1, 0, -\lambda_2, 0, -\lambda_3$ and with lower diagonal given by $\lambda_1, -1, \lambda_2, 1, \lambda_3$ (instead of -1 and 1 one can choose any negative and positive number respectively) and use this matrix as initial condition for the real form of the flow (5.1.1). What was observed numerically in [1] is that indeed, in this case, the flow converges to a real skew-symmetric tridiagonal matrix, isospectral to A_0 and having the same signs pattern for the codiagonal elements as the initial condition A_0. However, no proof of this fact was provided.

Proposition 5.18 *Using the flow* (5.1.1) *one can construct even dimensional real skew-symmetric tridiagonal matrices with simple imaginary spectrum and with given signs pattern for the codiagonal elements.*

Proof: Consider the flow (5.1.1) and suppose that the initial condition A_0 is a real upper Hessenberg matrix with given simple imaginary spectrum and with given signs pattern for the codiagonal elements, like the one constructed in the example above for a 6×6 matrix. First of all, observe that if $A(t)$ is real for some t then it remains real for all t. Secondly, since the flow (5.1.1) is isospectral, the ω-limit set $\Omega(A_0) \subset \mathcal{N}_\Lambda$, but real normal matrices with simple imaginary spectrum are necessarily skew-symmetric.

Therefore, by Theorem 1.1, $A(t)$ converges to a real skew-symmetric tridiagonal matrix A_∞ with nonzero codiagonal elements. Moreover, since all the entries in this case are real, equation (5.1.6) shows that the lower codiagonal elements can not change sign. Which means that the signs pattern for the lower codiagonal elements in A_∞ is the same as the signs pattern for A_0. ■

Chapter 6

Optimality of the Flow

6.1 Optimal Control Problem

Control theory deals with the systems that can be controlled. Here we consider a control system described by ordinary differential equations:

$$\dot{x} = f(t, x(t), u(t)) \qquad x(t_0) = x_0, \tag{6.1.1}$$

where x is the state in n dimensional smooth manifold, parameters $u \in \mathcal{U} \subset \mathbb{R}^m$ with control set $\mathcal{U}$, t_0 is the initial time and x_0 is the initial state. For each initial condition x_0 there are many trajectories depending on the choice of the control parameters u. There are two different ways to choose the control: open loop and closed loop. Choosing u as a function of time t is open loop and choosing u as a function of space variable x is closed loop.

The cost functional is key ingredient of an optimal control problem and will be of the form:

$$J(u(\cdot)) := \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(t_1, x) \tag{6.1.2}$$

where L and K are given functions known as running cost and terminal cost respectively. t_1 is the final time which is fixed or free and $x(t_1)$ is the final state. Problem is to find u which minimizes $J(u)$ over all admissible controls.

Suppose there is no terminal cost ($K \equiv 0$). We let the final time t_1 approach to ∞; in the limit set, the cost functional becomes:

$$J(u(\cdot)) := \int_{t_0}^{\infty} L(t, x(t), u(t))dt.$$

This is known as infinite time horizon problem.

6.2 The HJB Equation

We consider the cost functionals:

$$J(t, x, u) := \int_{t}^{t_1} L(s, x(s), u(s))ds + K(t_1, x)$$

where t ranges over $[t_0, t_1)$ and x ranges over $\mathbb{R}^n$, here $x(\cdot)$ on the right-hand side denotes the state trajectory corresponding to the control $u(\cdot)$ and satisfying $x(t) = x$.

Define value function:

$$V(t, x) = \inf_{u_{[t,t_1]}} J(t, x, u(\cdot))$$

where the notation $u_{[t,t_1]}$ indicates that the control u is restricted to the interval $[t, t_1]$.

By definition of the control problem, we have $V(t_1, x) = K(t_1, x)$. If there is no terminal cost ($K = 0$), then $V(t_1, x) = 0$.

The PDE

$$-V_t(t, x) = \inf_{u} \left\{ L(t, x, u(t)) + \langle V_x(t, x), f(t, x, u(t)) \rangle \right\}.$$

is called the HJB equation. To derive this for value function $V(t, x)$ we use Bellman's Principle of Optimality (or Principle of Dynamic Programming): for every (t, x) and every $\Delta t > 0$, we have

$$
\begin{aligned}
V(t,x) &= \inf_{u_{[t,t+\Delta t]}} J(t,x,u(\cdot)) \\
&= \inf_{u_{[t,t+\Delta t]}} \left\{ \int_t^{t+\Delta t} L(s,x(s),u(s))ds + V(t+\Delta t, x(t+\Delta t)) \right\}.
\end{aligned}
$$

Take Δt, write

$$x(t+\Delta t) = x + f(t,x,u(t))\Delta t + h.o.t.(\Delta t)$$

where $x(t) = x$ and $h.o.t.$ is higher order terms.

Taylor expansion of $V(t+\Delta t, x(t+\Delta t))$ gives us:

$$V(t+\Delta t, x(t+\Delta t)) = V(t,x) + V_t(t,x)\Delta t + \langle V_x(t,x), f(t,x,u(t))\Delta t\rangle + h.o.t.(\Delta t).$$

Also,

$$\int_t^{t+\Delta t} L(s,x(s),u(s))ds = L(t,x,u(t))\Delta t + h.o.t.(\Delta t).$$

Plug these into the Bellman's Principle of Optimality, cancel $V(t,x)$ from both sides because $V(t,x)$ does not depend on u and we get simplified expression as:

$$0 = \inf_{u_{[t,t+\Delta t]}} \left\{ L(t,x(t),u(t))\Delta t + V_t(t,x)\Delta t + \langle V_x(t,x), f(t,x,u(t))\Delta t\rangle + h.o.t.(\Delta t) \right\}.$$

Dividing both sides by Δt and letting $\Delta t \to 0$ implies higher order term disappear. Then, since inf is over the instantaneous values of u at time t and $V_t(t,x(t))$ doesn't depend on u. So taking it out from inf gives us HJB equation:

$$-V_t(t,x) = \inf_u \left\{ L(t,x,u(t)) + \langle V_x(t,x), f(t,x,u(t))\rangle \right\}.$$

6.3 Optimality of the Flow

Let us observe that equation (1.0.1) can be also viewed as a realization of the choice of a feedback for a controlled Lax system of the form

$$\frac{dA}{dt} = [U, A], \tag{6.3.1}$$

where matrix function $U = U(t)$ is the control input. Then one can view equation (6.3.1) as a general control system; to obtain a specific behavior, one has not only to select a admissible initial condition $A(0) := A_0$ but also to choose a specific feedback control law that substituted in place of $U(t)$ makes the system behave in a desired way.

We show that the system studied here is the solution of an infinite time horizon optimal control problem, using the Hamilton-Jacobi-Bellman approach.

Theorem 6.1 *Consider the following deterministic optimal control problem over an infinite time horizon:*

$$\begin{aligned} &\min_U \int_0^{+\infty} \operatorname{trace}\left([A^\dagger, A]_{du}([A^\dagger, A]_{du})^\dagger\right) + \operatorname{trace}(UU^\dagger)\, ds, \\ &\text{subject to } \frac{dA}{dt} = [U, A], \end{aligned} \tag{6.3.2}$$

where $U(t)$ is a sufficiently smooth function taking value in the Lie algebra of upper triangular matrices. Then the optimal value function is given by $V(A) = \operatorname{trace}(AA^\dagger)$ and the optimal feedback is given by $U = [A^\dagger, A]_{du}$, i.e., the flow (5.1.1) *is the solution of this infinite time horizon optimal control problem.*

Here U is assumed to be just upper triangular. The fact that U has zero trace is then a consequence of the form of the optimal solution.

Proof: The Hamilton-Jacobi-Bellman equation which determines the optimal

$U(t)$ for the problem above is given by

$$\min_U \left[\text{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger \right) + \frac{d}{dt}V(A) \right] = 0. \tag{6.3.3}$$

If the value function is smooth, the fulfillment of the above equation is a sufficient condition for optimality (see for instance [26]). Since $V(A) = \text{trace}(AA^\dagger)$ and using equation (6.3.1), we obtain

$$\frac{d}{dt}V(A) = 2\text{Re}(\text{trace}(\dot{A}A^\dagger)) = 2\text{Re}(\text{trace}([U, A]A^\dagger))$$

$$= 2\text{Re}(\text{trace}((U[A, A^\dagger])) = -2\text{Re}(\text{trace}(U[A^\dagger, A])).$$

Using this in (6.3.3) together with the fact that trace $\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger\right)$ is real, we have

$$\min_U \left[\text{Re}\left(\text{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger - 2U[A^\dagger, A] \right)\right)\right] = 0.$$

Moreover, since U is assumed to be an upper triangular matrix, we have that trace $\left(U[A^\dagger, A]_u)\right) = 0$. Hence,
$\text{Re}\left(\text{trace}\left(U[A^\dagger, A]\right)\right) = \text{Re}\left(\text{trace}\left(U([A^\dagger, A]_{dl} + [A^\dagger, A]_u)\right)\right) = \text{Re}\left(\text{trace}(U[A^\dagger, A]_{dl})\right)$ $= \text{Re}\left(\text{trace}\left((U[A^\dagger, A]_{dl})^\dagger\right)^\dagger\right) = \text{Re}\left(\text{trace}\left(([A^\dagger, A]_{dl})^\dagger U^\dagger\right)\right)$ (there is no conjugation because we are taking the real part only). Moreover, since $[A^\dagger, A]$ is Hermitian we have $([A^\dagger, A]_{dl})^\dagger = [A^\dagger, A]_{du}$. Therefore, the Hamilton-Jacobi-Bellman equation becomes

$$\min_U \left[\text{Re}\left(\text{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger - 2[A^\dagger, A]_{du}U^\dagger \right)\right)\right] = 0.$$

To find the optimal U we just take the gradient with respect to U of

$$\text{Re}\left(\text{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger - 2[A^\dagger, A]_{du}U^\dagger \right)\right),$$

and set it to zero. Observe that if A, B are any complex matrices, then $\mathrm{Re}(\mathrm{trace}(AB^\dagger))$ is the real part of the Hermitian inner product, which is known to be a real scalar product. Indeed, representing $A = A_0 + iA_1$ and $B = B_0 + iB_1$ where $A_0 = \mathrm{Re}(A)$ and $B_0 = \mathrm{Re}(B)$ and $A_1 = \mathrm{Im}(A)$ and $B_1 = \mathrm{Im}(B)$, a direct computation shows that

$$\mathrm{Re}(\mathrm{trace}(AB^\dagger)) = \mathrm{trace}(A_0B_0^T + A_1B_1^T) =$$

$$= \mathrm{trace}\left(\begin{pmatrix} A_0 & \mathbb{O} \\ \mathbb{O} & A_1 \end{pmatrix}\begin{pmatrix} B_0 & \mathbb{O} \\ \mathbb{O} & B_1 \end{pmatrix}^T\right) = \langle \begin{pmatrix} A_0 & \mathbb{O} \\ \mathbb{O} & A_1 \end{pmatrix}, \begin{pmatrix} B_0 & \mathbb{O} \\ \mathbb{O} & B_1 \end{pmatrix} \rangle,$$

where we have denoted here with $\langle \cdot, \cdot \rangle$ the scalar product induced on the corresponding real vector space.

We claim that $\nabla_U \mathrm{Re}((\mathrm{trace}(\mathrm{UU}^\dagger))) = 2U$ and $\nabla_U \left(\mathrm{Re}(\mathrm{trace}([A^\dagger, A]_{du}U^\dagger)\right) = [A^\dagger, A]_{du}$.

To prove the claim we use the definition of gradient and assume that we have a smooth parametrized curve $s \mapsto U(s)$ (here $U(s)$ is just a parametrized curve, and it has nothing to do with the evolution of the differential equation under study). Then $\frac{d}{ds}\mathrm{Re}\left(\mathrm{trace}(UU^\dagger)\right) = 2\mathrm{Re}\left(\mathrm{trace}(\dot{U}U^\dagger)\right) = 2\mathrm{Re}\left(\mathrm{trace}(U\dot{U}^\dagger)\right)$.

Writing $U = U_0 + iU_1 = \begin{pmatrix} U_0 & \mathbb{O} \\ \mathbb{O} & U_1 \end{pmatrix}$ and $\dot{U} = \dot{U}_0 + i\dot{U}_1 = \begin{pmatrix} \dot{U}_0 & \mathbb{O} \\ \mathbb{O} & \dot{U}_1 \end{pmatrix}$, we get $\frac{d}{dt}\mathrm{Re}(\left(\mathrm{trace}(UU^\dagger)\right)) = 2\mathrm{Re}(\mathrm{trace}(U\dot{U}^\dagger)) = 2\mathrm{trace}(U_0\dot{U_0}^T + U_1\dot{U_1}^T)$

$$= 2\mathrm{trace}\left(\begin{pmatrix} U_0 & \mathbb{O} \\ \mathbb{O} & U_1 \end{pmatrix}\begin{pmatrix} \dot{U_0}^T & \mathbb{O} \\ \mathbb{O} & \dot{U_1}^T \end{pmatrix}\right) = \mathrm{trace}\left(\begin{pmatrix} 2U_0 & \mathbb{O} \\ \mathbb{O} & 2U_1 \end{pmatrix}\begin{pmatrix} \dot{U}_0 & \mathbb{O} \\ \mathbb{O} & \dot{U}_1 \end{pmatrix}^T\right),$$

which is equal to $\langle \nabla_U \mathrm{Re}(\mathrm{trace}(UU^\dagger)), \dot{U} \rangle$, by definition of gradient. This gives

$$\nabla_U \mathrm{Re}(\mathrm{trace}(UU^\dagger)) = \begin{pmatrix} 2U_0 & \mathbb{O} \\ \mathbb{O} & 2U_1 \end{pmatrix} = 2U.$$

For other part of the claim, denote $[A^\dagger, A]_{du} = B$ then a similar straightforward calculation shows that $\nabla_U \left(\mathrm{Re}(\mathrm{trace}(BU^\dagger)\right) = B$.

Therefore the optimal U which is upper triangular is given by $[A^\dagger, A]_{du}$ and it is indeed a minimum since the expression

$$\mathrm{Re}\left(\mathrm{trace}\left(([A^\dagger, A]_{du})([A^\dagger, A]_{du})^\dagger + UU^\dagger - 2[A^\dagger, A]_{du}U^\dagger\right)\right)$$

is quadratic in U and convex. Furthermore, substituting $U = [A^\dagger, A]_{du}$ in the above expression yields zero identically and consequently $V(A) = \mathrm{trace}(AA^\dagger)$ is the value function and U is the optimal feedback. ∎

Another proof of

$$\nabla_U(\mathrm{trace}(\mathrm{UU}^\dagger)) = 2U,$$

and

$$\nabla_U \left(\mathrm{Re}(\mathrm{trace}([A^\dagger, A]_{du}U^\dagger)\right) = [A^\dagger, A]_{du}.$$

Following [23], if f is a real function of a complex matrix Z, then the complex gradient matrix is given by

$$\nabla f(Z) = 2\frac{df(Z)}{dZ^*} = \frac{\partial f(Z)}{\partial \mathrm{Re}(Z)} + i\frac{\partial f(Z)}{\partial \mathrm{Im}(Z)}.$$

Also,

$$\frac{\partial \mathrm{trace}(AX^\dagger)}{\partial \mathrm{Re}(X)} = A \text{ and } i\frac{\partial \mathrm{trace}(AX^\dagger)}{\partial \mathrm{Im}(X)} = A.$$

Let $U_{jk} = u_{jk} + iv_{jk}$ and $([A^\dagger, A]_{du})_{jk} = B_{jk} = \alpha_{jk} + i\beta_{jk}$, then

$$\mathrm{trace}([A^\dagger, A]_{du}U^\dagger) = \sum_{j=1}^{n}\sum_{k=1}^{n}\left((\alpha_{jk}u_{jk} + \beta_{jk}v_{jk}) + i(\beta_{jk}u_{jk} - \alpha_{jk}v_{jk})\right).$$

Also,

$$\mathrm{Re}\left(\mathrm{trace}([A^\dagger, A]_{du}U^\dagger)\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}(\alpha_{jk}u_{jk} + \beta_{jk}v_{jk}).$$

And,

$$\mathrm{trace}(UU^\dagger) = \sum_{j=1}^{n}\sum_{k=1}^{n}(u_{jk}^2 + v_{jk}^2).$$

Using these information we have

$$\nabla_U(\mathrm{trace(UU^\dagger)}) = 2U,$$

and

$$\nabla_U\left(\mathrm{Re}(\mathrm{trace}([A^\dagger, A]_{du}U^\dagger)\right) = [A^\dagger, A]_{du}.$$

Chapter 7

Simulations

In this chapter we briefly present some simulations illustrating the convergence properties of the flow.

We use the notation introduced in [11] in which three row vectors are used to describe tridiagonal normal matrices, the diagonal entries, lower codiagonal and upper codiagonal.

Example 7.1

First, we examine some simulations illustrating the convergence properties of the system introduced. We show that one can indeed generate an arbitrary nonzero elements in the codiagonal, choosing nonzero number in the lower diagonal elements of A_0. Suppose A_0 is a 7×7 lower bidiagonal matrix with diagonal entries given by $[1+i,\ 2+2i,\ 3+3i,\ -4-4i,\ 5+5i,\ -6-6i,\ 7+7i]$, with lower diagonal entries given by $[-1+i,\ -1+i,\ -1+i,\ 1+i,\ -1+i,\ 1+i]$. So, $A_0 =$

$$\begin{bmatrix}
1.0000 + 1.0000i & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
-1.0000 + 1.0000i & 2.0000 + 2.0000i & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -1.0000 + 1.0000i & 3.0000+3.0000i & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -1.0000 + 1.0000i & -4.0000-4.0000i & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 1.0000 + 1.0000i & 5.0000+5.0000i & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 + 1.0000i & -6.0000-6.0000i & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 + 1.0000i & 7.0000+7.0000i
\end{bmatrix},$$

and with spectrum $\sigma(A_0) = \{1+i,\ 2+2i,\ 3+3i,\ -4-4i,\ 5+5i,\ -6-6i,\ 7+7i\}$ $\subset l \subset \mathbb{C}$ and corresponding ω-limit point: $\Omega(A_0) =$

$$\begin{bmatrix} \text{1.3685 + 1.3685i} & \text{0.5494 - 0.5495i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\ \text{-0.5495 + 0.5494i} & \text{2.0016 + 2.0016i} & \text{0.5699 - 0.5694i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\ \text{0.0000} & \text{-0.5694 + 0.5699i} & \text{2.2346 + 2.2346i} & \text{1.6056 - 1.6083i} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\ \text{0.0000} & \text{0.0000} & \text{-1.6083 + 1.6056i} & \text{-3.4647 - 3.4647i} & \text{1.0445 + 1.0457i} & \text{0.0000} & \text{0.0000} \\ \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{1.0457 + 1.0445i} & \text{4.7637 + 4.7637i} & \text{1.0130 - 1.0134i} & \text{0.0000} \\ \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{-1.0134 + 1.0130i} & \text{-5.8232 - 5.8232i} & \text{0.9961 + 0.9962i} \\ \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.9962 + 0.9961i} & \text{6.9196 + 6.9196i} \end{bmatrix}.$$

Using our flow, a numerical approximation of the corresponding ω-limit point is a normal tridiagonal matrix $\Omega(A_0)$ with diagonal entries given by $[1.3685 + 1.3685i,$ $2.0016 + 2.0016i,\ 2.2346 + 2.2346i, -3.4647 - 3.4647i,\ 4.7637 + 4.7637i, -5.8232 - 5.8232i, 6.9196 + 6.9196i]$, with lower codiagonal entries given by $[-0.5495 + 0.5494i,$ $-0.5694 + 0.5699i,\ -1.6083 + 1.6056i, 1.0457 + 1.0445i, -1.0134 + 1.0130i, 0.9962 + 0.9961i]$. Also, upper codiagonal entries are given by $[0.5494 - 0.5495i, 0.5699 - 0.5694i,\ 1.6056 - 1.6083i, 1.0445 + 1.0457i,\ 1.0130 - 1.0134i, 0.9961 + 0.9962i]$.

The spectrum $\sigma(\Omega(A_0)) = \{0.9999 + 0.9999i, 2.0000 + 2.0000i, 3.0000 + 3.0000i,$ $-3.9999 - 3.9999i, 5.0000 + 5.0000i, -5.9999 - 5.9999i, 7.0000 + 7.0000i\}$ is within the third decimal digit from the spectrum of A_0. It can be seen that the nonzero pattern and absolute value of corresponding codiagonal elements has been faithfully reproduced.

Example 7.2

Secondly, let us present an example with real spectrum. Suppose A_0 is a 7×7 lower bidiagonal matrix with diagonal entries given by $[1,\ -2,\ 4,\ 2,\ 5,\ -6,\ 8]$, with lower diagonal entries are given by $[-1-i,\ 1+i,\ -1+i,\ 1+i,\ -3+i,\ 1+2i]$. So, $A_0 =$

$$\begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.0000 - 1.0000i & -2.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 + 1.0000i & 4.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 + 1.0000i & 2.000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 + 1.0000i & 5.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -3.0000 + 1.0000i & -6.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 + 2.0000i & 8.0000 \end{bmatrix},$$

and with spectrum $\sigma(A_0) = \{1,\ -2,\ 4,\ 2,\ 5,\ -6,\ 8\}$. Corresponding ω-limit point: $\Omega(A_0) =$

$$\begin{bmatrix} 0.7320 - 0.0000i & -0.7629 + 0.7629i & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.7629 - 0.7629i & -0.8725 - 0.0000i & 1.2969 - 1.2968i & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.2969 + 1.2968i & 2.9959 - 0.0000i & -0.6811 - 0.6811i & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.6811 + 0.6811i & 2.7206 - 0.0000i & 0.7606 - 0.7606i & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.7606 + 0.7606i & 3.0104 - 0.0000i & -3.5803 - 1.1932i & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -3.5803 + 1.1932i & -4.0045 + 0.0000i & 1.0243 - 2.0485i \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0243 + 2.0485i & 7.4181 - 0.0000i \end{bmatrix}.$$

Using our flow, a numerical approximation of the corresponding ω-limit point is a normal tridiagonal matrix $\Omega(A_0)$ with diagonal entries given by $[0.7320, -0.8725, 2.9959, 2.7206,\ 3.0104, -4.0045, 7.4181]$, and with lower codiagonal entries given by $[-0.7629 - 0.7629i, 1.2969 + 1.2968i,\ -0.6811 + 0.6811i, 0.7606 + 0.7606i, -3.5803 + 1.1932i, 1.0243 + 2.0485i]$. Also, upper codiagonal entries are given by $[-0.7629 + 0.7629i, 1.2969-1.2968i, -0.6811-0.6811i, 0.7606-0.7606i, -3.5803-1.1932i, 1.0243-2.0485i]$.

The spectrum $\sigma(\Omega(A_0)) = \{0.9999, -2.0000, 4.0000, 2.0000, 4.9999, -6.0000, 8.0000\}$ is within the third decimal digit from the spectrum of A_0. It can be seen that Hermitian matrix is reproduced with faithful nonzero pattern of the codiagonal elements.

Example 7.3

Thirdly, we consider the case of pure imaginary spectrum. Suppose A_0 is a 7×7 lower bidiagonal matrix with diagonal entries given by $[i,\ 2i,\ 3i,\ -4i,\ 5i,\ -6i,\ -7i]$, with lower diagonal entries given by $[-1+i,\ -1+2i,\ -1+i,\ 4+i,\ -1+i,\ 1]$. So, $A_0 =$

$$\begin{bmatrix}
\text{1.0000i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{-1.0000 + 1.0000i} & \text{2.0000i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{-1.0000 + 2.0000i} & \text{3.0000i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{-1.0000 + 1.0000i} & \text{-4.000i} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{4.0000 + 1.0000i} & \text{5.0000i} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{-1.0000 + 1.0000i} & \text{-6.0000i} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{1.0000} & \text{-7.0000i}
\end{bmatrix},$$

with spectrum $\sigma(A_0) = \{i,\ 2i,\ 3i,\ -4i,\ 5i,\ -6i,\ -7i\}$. Corresponding ω-limit point: $\Omega(A_0) =$

$$\begin{bmatrix}
\text{0.0000 + 1.5957i} & \text{0.4670 + 0.4670i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{-0.4670 + 0.4670i} & \text{0.0000 + 2.0312i} & \text{0.3654 + 0.7309i} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{-0.3654 + 0.7309i} & \text{-0.0000 + 1.2751i} & \text{2.1518 + 2.1518i} & \text{0.0000} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{-2.1518 + 2.1518i} & \text{-0.0000 - 1.7487i} & \text{-2.3674 + 0.5918i} & \text{0.0000} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{2.3674 + 0.5918i} & \text{0.0000 + 2.8350i} & \text{1.9094 + 1.9094i} & \text{0.0000} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{-1.9094 + 1.9093i} & \text{0.0000 - 5.2726i} & \text{-0.4940 + 0.0000i} \\
\text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.0000} & \text{0.4940 - 0.0000i} & \text{0.0000 - 6.7157i}
\end{bmatrix}.$$

Using our flow, a numerical approximation of the corresponding ω-limit point is a normal tridiagonal matrix $\Omega(A_0)$ with diagonal entries given by $[1.5957i, 2.0312i, 1.2751i, -1.7487i, 2.8350i, -5.2726i, -6.7157i]$ and with lower codiagonal entries given by $[-0.4670+0.4670i, -0.3654+0.7309i, -2.1518+2.1518i, 2.3674+0.5918i, -1.9094+1.9093i, 0.4940-0.0000i]$. Also, upper codiagonal entries are given by $[0.4670+0.4670i, 0.3654+0.7309i, 2.1518+2.1518i, -2.3674+0.5918i, 1.9094+1.9094i, -0.4940+0.0000i]$.

The spectrum $\sigma(\Omega(A_0)) = \{0.9999i, 1.9999i, 3.0000i, -4.0002i, 5.0002i, -5.9999i, -6.9999i\}$ is within the third decimal digit from the spectrum of A_0. It can be seen that skew-Hermitian matrix is reproduced with the nonzero pattern of the codiagonal elements.

Example 7.4

Now, we give an example showing if simple spectrum does not lie on a line, flow does not converge to tridiagonal matrix. Consider, a matrix A_0 with spectrum lying on the unit circle centered at the origin. Suppose A_0 is a 7×7 lower bidiagonal matrix with diagonal entries given by $[i,\ -i,\ 1,\ -1,\ -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}},\ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}},\ \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}]$ and with lower diagonal entries given by $[1-i,\ -1-i,\ -1+i,\ 1+i,\ -1+i,\ 1+i]$. Let $A_0 =$

$$\begin{bmatrix} 0 + 1.0000i & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 - 1.0000i & 0 - 1.0000i & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 - 1.0000i & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 + 1.0000i & -1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 + 1.0000i & -0.7071 + 0.7071i & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 + 1.0000i & 0.7071 - 0.7071i & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 + 1.0000i & 0.7071+0.7071i \end{bmatrix},$$

with spectrum $\sigma(A_0) = \{i, -i, 1, -1, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\}$. A numerical approximation of the corresponding ω-limit point using our flow is given by: $\Omega(A_0) =$

$$\begin{bmatrix} 0.0630 + 0.3639i & -0.0433 - 0.3068i & -0.0291 - 0.0262i & 0.3953 + 0.2295i & -0.0668 - 0.0954i & 0.3471 - 0.1956i & -0.4384 + 0.4385i \\ 0.6570 - 0.6570i & 0.0898 - 0.0844i & 0.0018 - 0.0154i & 0.0160 + 0.1811i & 0.0157 - 0.0435i & 0.1431 + 0.0680i & -0.2429 - 0.0420i \\ 0.0000 & -0.6665 - 0.6666i & 0.0148 + 0.0012i & -0.1728 + 0.0206i & 0.0421 + 0.0137i & -0.0608 + 0.1390i & 0.0330 - 0.2338i \\ 0.0000 & 0.0000 & -0.7063 + 0.7064i & -0.0128 + 0.0195i & 0.0051 - 0.0031i & 0.0090 + 0.0182i & -0.0211 - 0.0236i \\ 0.0000 & 0.0000 & 0.0000 & 0.6029 + 0.6030i & 0.0280 + 0.0765i & -0.2699 + 0.0700i & 0.3754 - 0.2180i \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.6984 + 0.6984i & 0.0535 + 0.0651i & -0.1074 - 0.0752i \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.5948 + 0.5949i & 0.4709 + 0.2654i \end{bmatrix},$$

which is a normal upper Hessenberg matrix isospectral to A_0 but not a tridiagonal matrix.

Example 7.5

Now, we present an example to construct an even dimensional real skew-symmetric tridiagonal matrix with given imaginary spectrum and given sign pattern of lower codiagonal elements. For example the spectrum is $\pm i, \pm 3i, \pm\sqrt{2}i, \pm 4i$ and lower codiagonal signs pattern is $+, -, +, +, +, -, -$. Then choose initial condition A_0 as:

$$A_0 = \begin{bmatrix} 0.0000 & -1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 & -3.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & -1.4142 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.4142 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 & 0.0000 & 4.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -4.0000 & 0.0000 \end{bmatrix}.$$

A numerical approximation of the corresponding ω-limit point using our flow is given by:

$$\Omega(A_0) = \begin{bmatrix} 0.0000 & -1.0827 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0827 & 0.0000 & 0.7826 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.7826 & 0.0000 & -2.7384 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 2.7384 & 0.0000 & -0.8289 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.8289 & 0.0000 & -1.4986 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.4986 & 0.0000 & 1.0929 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0929 & 0.0000 & 3.8196 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -3.8196 & 0.0000 \end{bmatrix},$$

which is a real skew-symmetric tridiagonal matrix, isospectral to A_0 and having the same signs pattern for the codiagonal elements as the initial condition A_0.

Example 7.6

Finally, we present a lower dimensional case (a 4×4 matrix), in which we display the transient behavior of the lower codiagonal elements. In our example the matrix A_0 is a 4×4 real matrix, with vanishing diagonal elements, with upper codiagonal elements given by $[-1, -2, -3]$ and lower codiagonal elements given by $[1, 0.8, 1.2]$ and all the other entries are zero.

$$A_0 = \begin{bmatrix} 0.0000 & -1.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & -2.0000 & 0.0000 \\ 0.0000 & 0.8000 & 0.0000 & -3.0000 \\ 0.0000 & 0.0000 & 1.2000 & 0.0000 \end{bmatrix}.$$

This matrix has purely imaginary spectrum given by $\{\pm 0.8053i, \pm 2.3562i\}$. The flow converges to the following approximation of the ω-limit point: a 4×4 tridiagonal skew-symmetric real matrix, with vanishing diagonal elements and lower codiagonal elements given by $[0.9545, 1.1566, 1.9883]$ with spectrum given by $\{\pm 0.8053i, \pm 2.3566i\}$.

$$\Omega(A_0) = \begin{bmatrix} 0.0000 & -0.9545 & 0.0000 & 0.0000 \\ 0.9545 & 0.0000 & -1.1566 & 0.0000 \\ 0.0000 & 1.1566 & 0.0000 & -1.9883 \\ 0.0000 & 0.0000 & 1.9883 & 0.0000 \end{bmatrix}.$$

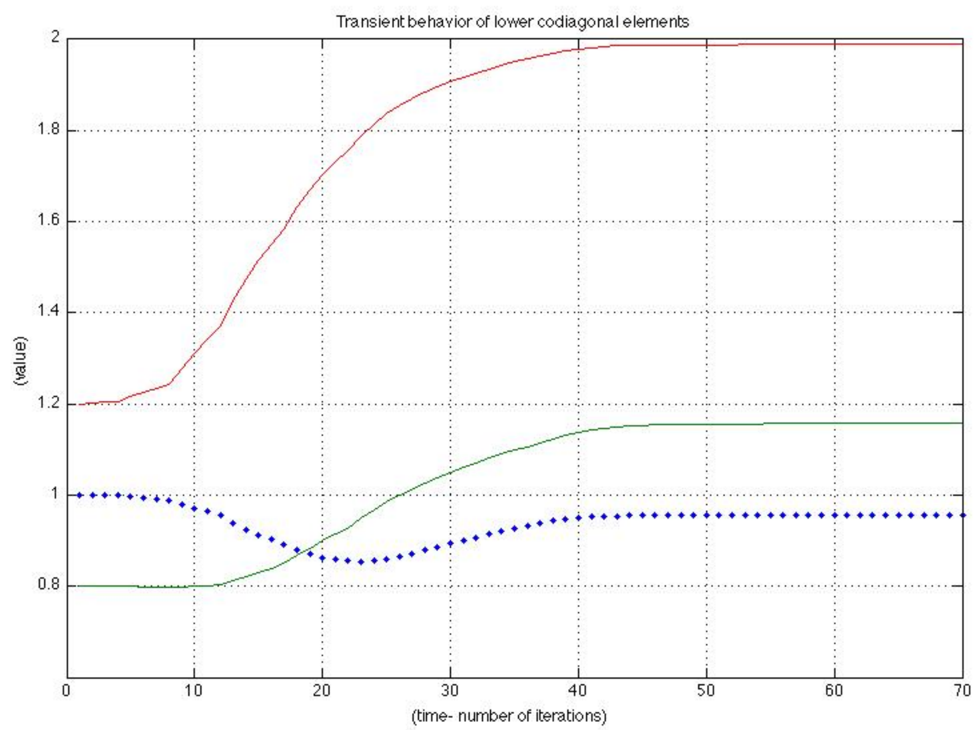

Figure 7-1: Transient behavior of lower codiagonal elements

The transient behavior of the lower codiagonal elements is given in the Figure 7-1 above.

The transient behavior of the lower codiagonal elements in Example 7.5 is given

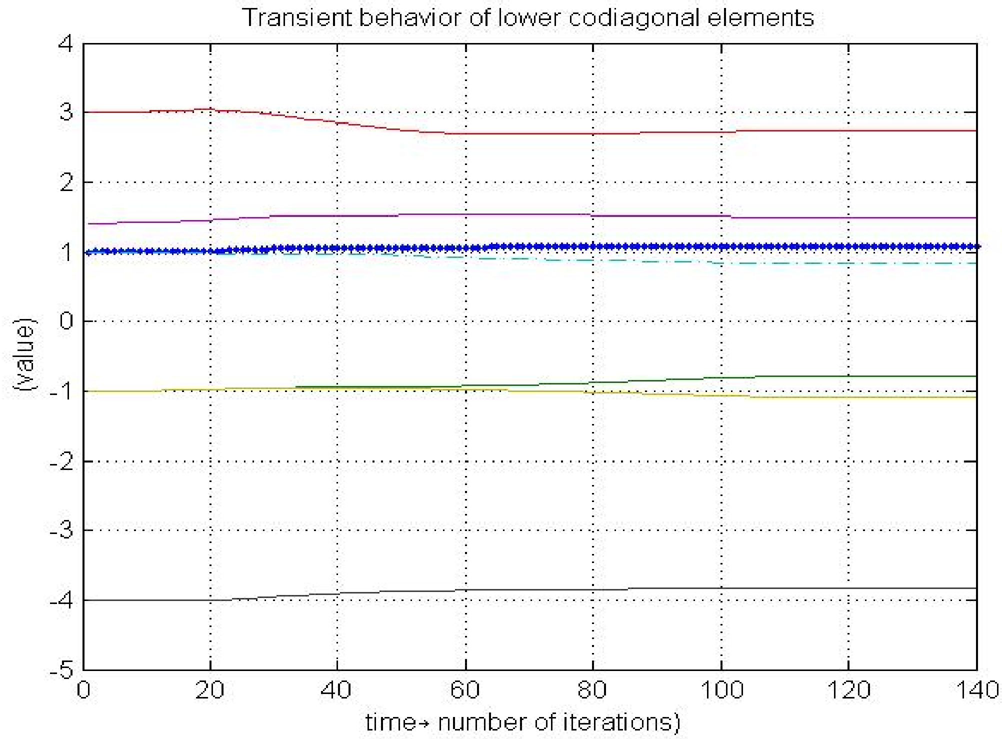

Figure 7-2: Transient behavior of lower codiagonal elements in Example 7.5

in the Figure 7-2.

We performed several numerical experiments in other cases and in general we observed that there seems to be a less oscillatory transient behavior for the codiagonal elements compared to what was observed in the examples treated in [30].

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Appendix A

The Detailed Proof of Proposition 4.3

Proof: Consider flow 4.1.7 and let $A(t) = T(t)A_0T(t)^{-1}$. Let the eigenvalues of A_0 are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since, $A(t)$ and A_0 are similar, they have the same eigenvalues. Therefore,

$$\mathrm{trace}(A(t)) = \mathrm{trace}(A_0)$$

Similarly, $A^2(t) = T(t)A_0T(t)^{-1}T(t)A_0T(t)^{-1} = T(t)A_0^2T(t)^{-1}$. Since, $A^2(t)$ and A_0^2 are similar, they have the same eigenvalues. Therefore,

$$\mathrm{trace}(A^2(t)) = \mathrm{trace}(A_0^2)$$

and so on up to $\mathrm{trace}(A^n(t)) = \mathrm{trace}(A_0^n)$.

Now,

$$\mathrm{trace}(A(t)) = \mathrm{trace}(A_0) \Longrightarrow \lambda_1 + \lambda_2 + \ldots + \lambda_n := f_1$$

$$\mathrm{trace}(A^2(t)) = \mathrm{trace}(A_0^2) \Longrightarrow \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 := f_2$$

and so on up to

$$\text{trace}(A^n(t)) = \text{trace}(A_0^n) \Longrightarrow \lambda_1^n + \lambda_2^n + \ \dots \ + \lambda_n^n := f_n$$

Note that if $\lambda_1, \lambda_2, \ \dots \ , \lambda_n$ are eigenvalues of A then $\lambda_1^k, \lambda_2^k, \ \dots \ , \lambda_n^k$ are eigenvalues of A^k.

Here, f_k are polynomial on n variables $\lambda_1^k, \lambda_2^k, \ \dots \ , \lambda_n^k$.

Recall : (1) Elementary Symmetric Polynomials

$\sigma_k(x_1, \ \dots \ , x_n)$ (sometimes denoted by Π_k or e_λ) of n variables $(x_1, \ \dots \ , x_n)$ are defined by

$$\sigma_1(x_1, \ \dots \ , x_n) = \sum_{1 \le j \le n} x_j$$

$$\sigma_2(x_1, \ \dots \ , x_n) = \sum_{1 \le j < l \le n} x_j x_l$$

$$\sigma_3(x_1, \ \dots \ , x_n) = \sum_{1 \le j < l < p \le n} x_j x_l x_p$$

and so on

$$\sigma_n(x_1, \ \dots \ , x_n) = \Pi_{1 \le j \le n} x_j.$$

For example Elementary Symmetric Polynomials of four variables is

$$\sigma_1(x_1, \ \dots \ , x_n) = x_1 + x_2 + x_3 + x_4$$

$$\sigma_2(x_1, \ \dots \ , x_n) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

$$\sigma_3(x_1, \ \dots \ , x_n) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

$$\sigma_4(x_1, \ \dots \ , x_n) = x_1x_2x_3x_4.$$

Hence, considering n variables $(x_1, \ \dots \ , x_n)$ as $(\lambda_1, \ \dots \ , \lambda_n)$, we get σ_k are functions of $(\lambda_1, \lambda_2, \ \dots \ , \lambda_n)$ in different form.

(2) The Power Sum

$S_p(x_1, \ \ldots, x_n)$ is defined by $S_p(x_1, \ \ldots, x_n) = \sum_{j=1}^{n} x_j^p$. That is

$$S_1(x_1, \ \ldots, x_n) = x_1 + \ \ldots \ + x_n$$

$$S_2(x_1, \ \ldots, x_n) = x_1^2 + \ \ldots \ + x_n^2$$

and so on up to

$$S_n(x_1, \ \ldots, x_n) = x_1^n + \ \ldots \ + x_n^n$$

The relationship between S_p and σ_p is given by *Newton-Girard Formula.*

The *relation function* $s_p(\sigma_1, \ \ldots, \sigma_n)$ with arguments given by the Elementary Symmetric Polynomials (*not* x_n) is defined by

$$s_p(\sigma_1, \ \ldots, \sigma_n) = (-1)^{p-1} S_p(x_1, \ \ldots, x_n) = (-1)^{p-1} \sum_{j=1}^{n} x_j^p.$$

It turns out that $s_p(\sigma_1, \ \ldots, \sigma_n)$ is given by the coefficients of the generating function

$$\ln(1+\sigma_1 t+\sigma_2 t^2+\ldots) = \sum_{j=1}^{\infty} \frac{s_j}{j} t^j = \sigma_1 t + \frac{1}{2}(-\sigma_1^2+2\sigma_2)t^2 + \frac{1}{3}(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3)t^2 + \ldots$$

So the first few values are

$$s_1 = \sigma_1$$

$$s_2 = -\sigma_1^2 + 2\sigma_2$$

$$s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$

$$s_4 = -\sigma_1^4 + 4\sigma_1^2\sigma_2 - 2\sigma_2^2 - 4\sigma_1\sigma_3 + 4\sigma_4$$

In general, s_p can be computed from the determinant

$$s_p(\sigma_1, \ \ldots, \sigma_n) = (-1)^{p-1} \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & . \ . \ . & 0 \\ 2\sigma_1 & \sigma_1 & 1 & 0 & . \ . \ . & 0 \\ 3\sigma_3 & \sigma_2 & \sigma_1 & 1 & . \ . \ . & 0 \\ 4\sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 & . \ . \ . & 0 \\ . & . & . & . & . \ . \ . & . \\ . & . & . & . & . \ . \ . & . \\ . & . & . & . & . \ . \ . & . \\ p\sigma_p & \sigma_{p-1} & \sigma_{p-2} & \sigma_{p-3} & . \ . \ . & \sigma_1 \end{vmatrix}$$

(by Littlewood 1958, Cadogan 1971).

In particular,

$$S_1(x_1, \ \ldots, x_n) = x_1 + \ \ldots \ + x_n = \sigma_1$$

$$S_2(x_1, \ \ldots, x_n) = \sigma_1^2 - 2\sigma_2$$

$$S_3(x_1, \ \ldots, x_n) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$

$$S_4(x_1, \ \ldots, x_n) = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4$$

(by Schroeppel 1972).

(3) Characteristic Polynomials

Using the language of exterior algebra, one may completely express the characteristic polynomials of $n \times n$ matrix by

$$P_A(\lambda) = \lambda^n + \sum_{j=1}^{n} \lambda^{n-j}(-1)^j \mathrm{trace}(\Lambda^j A)$$

where $\mathrm{trace}(\Lambda^j A)$ is the trace of the j th exterior power of A with dimension $\binom{n}{j}$

and may be evaluate explicitly as the determinant of the $j \times j$ matrix

$$\frac{1}{j!}\begin{vmatrix} \text{trace}(A) & \text{j-1} & 0 & 0 & . & . & . & 0 \\ \text{trace}(A^2) & \text{trace}(A) & \text{j-2} & 0 & . & . & . & 0 \\ \text{trace}(A^3) & \text{trace}(A^2) & \text{trace}(A) & \text{j-3} & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \text{trace}(A^{j-1}) & \text{trace}(A^{j-2}) & \text{trace}(A^{j-3}) & . & . & . & . & 1 \\ \text{trace}(A^{j}) & \text{trace}(A^{j-1}) & \text{trace}(A^{j-2}) & . & . & . & . & \text{trace}(A) \end{vmatrix}$$

For example, the characteristic polynomials of 2×2 matrix A is given by

$$P_A(\lambda) = \lambda^2 + \sum_{j=1}^{2} \lambda^{2-j}(-1)^j \text{trace}(\Lambda^j A) = \lambda^2 - \lambda \text{trace}(\Lambda A) + \text{trace}(\Lambda^2 A)$$

Therefore,

$$P_A(\lambda) = \lambda^2 - \lambda \text{trace}(A) + \det(A).$$

Another way to represent characteristic polynomial is

$$P_A(\lambda) = \det(\lambda \mathbb{I} - A) = \lambda^n - \text{trace}(A)\lambda^{n-1} + \; .... \; + (-1)^{n-1}\det(A).$$

Now moving to the proof of the proposition, let $A(t)$ is the solution of (4.1.7), the characteristic polynomial of $A(t)$ is

$$P_A(\lambda) = \lambda^n - \text{trace}(A)\lambda^{n-1} + \; .... \; + (-1)^{n-1}\det(A) = \lambda^n - \sigma_1\lambda^{n-1} + \; .... \; + (-1)^{n-1}\sigma_n.$$

If $\frac{d}{dt}\left(\text{trace}(A^k)\right) = 0$ then σ_k is constant. But σ_k are coefficient of characteristic polynomial, which says that characteristic polynomial do not change with time.

To check $\frac{d}{dt}\left(\operatorname{trace}(A^k)\right) = 0$ if (4.1.7) is satisfied

$$\begin{aligned}
\frac{d}{dt}\left(\operatorname{trace}(A^k)\right) &= \frac{d}{dt}\left(\operatorname{trace}(A \cdot A \cdot A \cdots A)\right) \\
&= \operatorname{trace}(\dot{A} \cdot A^{k-1}) + \operatorname{trace}(A \cdot \dot{A} \cdot A^{k-2}) + \cdots + \operatorname{trace}(A^{k-1} \cdot \dot{A}) \\
&= \operatorname{trace}(\dot{A} \cdot A^{k-1}) + \operatorname{trace}(\dot{A} \cdot A^{k-1}) + \cdots + \operatorname{trace}(\dot{A} \cdot A^{k-1})
\end{aligned}$$

using the property $\operatorname{trace}(A \cdot B) = \operatorname{trace}(B \cdot A)$. Hence,

$$\frac{d}{dt}\left(\operatorname{trace}(A^k)\right) = k \cdot \operatorname{trace}(\dot{A} \cdot A^{k-1})$$

Now using the flow (4.1.7) we get

$$\frac{d}{dt}\left(\operatorname{trace}(A^k)\right) = k \cdot \operatorname{trace}([f(t,B),A]A^{k-1}) = k \cdot \operatorname{trace}(f(t,B)[A,A^{k-1}]) = 0$$

since $\operatorname{trace}([A,B]C) = \operatorname{trace}(A[B,C])$ and $[A,A^{k-1}] = 0$.

The most important thing here is that f_k are polynomial on n variables $\lambda_1^k, \lambda_2^k, \ \ldots \ , \lambda_n^k$ and σ_k are functions of $(\lambda_1, \lambda_2, \ \ldots \ , \lambda_n)$ and there is Newton's Identity which can change f_k to σ_k.

Since f_k are constant implies that σ_k are constant, characteristic polynomial does not change in time. This proves the proposition. $\blacksquare$

Appendix B

MATLAB Programmings

Here we present MATLAB programming codes used in Chapter 7 to discuss the behavior of the flow of the vector field (5.1.1).

Code for differential equation

```
function dx = model_myflowH2(t,x)
global n;
%Vector to matrix
for j=1:n
A(:,j)=x((j-1)*n+1:j*n);
end
U=triu(bracket(A',A));
dA=bracket(U,A);
%Matrix to vector
dx=dA(:);
```

The programming First Code is for Example 7.1, Second Code is for Example 7.4 and Third Code is for Example 7.6.

First Code

```
% Complex Hessenberg flow to show the convergence property
clear all;
```

```
global n;
% Dimension of the matrix
n=7;
% Initial condition
d_1=[-1+i -1+i -1+i 1+i -1+i 1+i];
d_0=[1+i 2+2*i 3+3*i -4-4*i 5+5*i -6-6*i 7+7*i];
A0=diag(d_0)+diag(d_1,-1)
%display eigenvalues of A0
eig(A0)
% Integration
x0=A0(:); [t,x]=ode15s(@model_myflowH2,[0 500],x0);
for j=1:n
A(:,j)=x(end,(j-1)*n+1:j*n);
end
%display ω-limit set of A0
A
%display eigenvalues of A
eig(A)
```

Second Code

```
%Complex Hessenberg flow to show the fifth point of Lemma 5.3 is not true in the case of general simple spectrum(i.e., if spectrum is not contained in a line)
clear all;
global n;
% Dimension of the matrix
n=7;
% Initial condition
d_1=[1-1*i -1-1*i -1+1*i 1+1*i -1+1*i 1+1*i];
```

```
d_0=[1*i -1*i 1 -1 -sqrt(1/2)+sqrt(1/2)*i sqrt(1/2)-sqrt(1/2)*i sqrt(1/2)+sqrt(1/2)*i];
A0=diag(d_0)+diag(d_1,-1)
%display eigenvalues of A0
eig(A0)
% Integration
x0=A0(:);
[t,x]=ode15s(@model_myflowH2,[0 500],x0);
for j=1:n
A(:,j)=x(end,(j-1)*n+1:j*n);
end
%display ω-limit set of A0
A
%display eigenvalues of A
eig(A)
```

Third Code

%Complex Hessenberg flow to construct an even dimensional real skew-symmetric tridiagonal matrix and display the transient behavior of the lower codiagonal elements

```
clear all;
global n;
% Dimension of the matrix
n=4;
% Initial condition
d_2=[-1 -2 -3];
d_1=[1 0.8 1.2];
d_0=[0 0 0 0];
A0=diag(d_0)+diag(d_1,-1)+diag(d_2,1)
%display eigenvalues of A0
```

```
eig(A0)
% Integration
for k=1:80
for j=1:n
x0=A0(:);
[t,x]=ode15s(@model_myflowH2,[0 k],x0);
A(:,j,k)=x(k,(j-1)*n+1:j*n);
end
%display subdiagonal element on each iterations
y(:,k,1)=A(2,1,k);
y(:,:,1)=y(:,:,1)
y(:,k,2)=A(3,2,k);
y(:,:,2)=y(:,:,2)
y(:,k,3)=A(4,3,k);
y(:,:,3)=y(:,:,3)
%plot transient behavior of the lower codiagonal elements
plot(1:k,y(:,:,1),'.',1:k,y(:,:,2),'-',1:k,y(:,:,3),'-')
%display grid on graph
grid on
end
%display ω-limit set of A0
A
%display eigenvalues of A
eig(A(:,:,k))
```