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Invariant Riemannain metrics on four-dimensional Lie group

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A Dissertation

entitled

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by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics

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An Abstract of

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This dissertation concerns invariant metrics on four-dimensional Lie groups. Various background material such as the automorphism group of a Lie algebra and the space of derivations is reviewed. The meaning of unimodularity for Lie algebras is discussed and the limitations of assuming that the metric is written in orthonormal form are investigated. Various curvature formulas for invariant metrics on Lie groups are obtained and the case where the Lie algebra has a codimension one abelian ideal is studied in detail. For each of the four-dimensional Lie groups the space of derivations of the associated Lie algebra is determined. The results are used to reduce the form of a right-invariant metric and the Ricci tensor is calculated. It is then possible to say precisely which metrics are Einstein spaces. In the indecomposable case precisely four such Einstein spaces are obtained. Two of them are spaces of constant curvature. Only one trivial example is found in the decomposable case.

To my parents Nara Bahadur Karki and Gaura Devi Karki

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Contents

Chapter 1

Preliminary Material

1.1 Introduction

In this dissertation I shall consider some curvature properties of invariant metrics on four-dimensional Lie groups. In order to formulate the results of my investigations it is necessary to present various background material. In this Chapter I will consider the automorphism group $Aut(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ which itself is a Lie group. The Lie algebra of $Aut(\mathfrak{g})$ consists of the space of derivations of $\mathfrak{g}$ as I show. I will also discuss the meaning and significance of uni-modularity for Lie algebras. It turns out that unimodularity is a more important property in three rather than four dimensions. After that I will consider the difficulty of assuming that the metric is written in orthonormal form. At the end of this Chapter I give the definition of an Einstein space and explain how they arise in the theory of general relativity.

In Chapter 2 various curvature formulas for invariant metrics on Lie groups are obtained and the case where the Lie algebra has a codimension one abelian ideal is studied in detail. It is shown by quoting results from [Tho] that in the solvable case we obtain an Einstein space if and only if the space is of constant curvature.

In Chapter 3 for each of the four-dimensional Lie groups the space of derivations of the associated Lie algebra is determined. We then exponentiate to find $Aut(\mathfrak{g})$. The results are used to reduce the form of a right-invariant metric and the Ricci tensor is calculated using the formulas that were obtained in Chapter 2. Although Ricci is much simpler than for a general four-dimensional algebra it still very complicated as the reader can see from the formulas displayed in Chapter 3.

At that point it becomes feasible to set the Ricci tensor of g equal to a multiple of g so as to say precisely which metrics are Einstein spaces. The corresponding Lie algebra $\mathfrak{g}$ is solvable in the indecomposable case because there are no semi-simple algebras in dimension four and hence, according to a Theorem of Milnor, [Mil], the Ricci tensor is negative definitive and the Lie group is non-compact. In fact the function of proportionality between the metric and the Ricci tensor is $\frac{\rho}{n}$ where ρ denotes the scalar curvature and n the dimension of the Lie group. We examine each possible four-dimensional algebra, first of all considering indecomposable then decomposable algebras. In the indecomposable case we find precisely four Einstein spaces, two of which are constant curvature spaces: one of the not constant curvature Einstein spaces depends upon a parameter that does not correspond to gauge freedom. We do not find any interesting cases in the decomposable case but it is done for the sake of completeness. Only one trivial example is found in the decomposable case.

We denote the Ricci tensor as a type $(1,1)$-tensor field by R or $Ricci(1,1)$ and the Ricci tensor as a type $(0,2)$-tensor field by R or $Ricci(0,2)$. Finally as regards notation, in the sequel we shall use the summation convention on repeated indices one a subscript and one a superscript wherever it is practical to do so.

1.2 Automorphism of a Lie Algebra

We consider a real finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. An *automorphism* Φ of $\mathfrak{g}$ is an invertible linear map from $\mathfrak{g}$ to $\mathfrak{g}$ such that for all $X, Y \in \mathfrak{g}$

$$\Phi([X, Y]) = [\Phi(X), \Phi(Y)]. \tag{1.1}$$

The set of all automorphisms of $\mathfrak{g}$ is denoted by $Aut(\mathfrak{g})$ and it is actually a Lie group: in fact it is a closed subgroup of $GL(\mathfrak{g})$. We shall now find its Lie algebra.

Proposition 1.2.1. *The Lie algebra of $Aut(\mathfrak{g})$ consists precisely of the space of derivations of $\mathfrak{g}$.*

Proof. We take a curve Φ_t in $Aut(\mathfrak{g})$ such that $\Phi_0 = \mathrm{id}_{\mathfrak{g}}$ and suppose that $X, Y \in \mathfrak{g}$. Then

$$\frac{d}{dt}\Phi_t[X, Y] = [\frac{d}{dt}\Phi_t X, \Phi_t Y] + [\Phi_t X, \frac{d}{dt}\Phi_t Y].$$

Putting $\frac{d}{dt}\Phi_t|_{t=0} = D$ gives

$$D[X, Y] = [DX, Y] + [X, DY]. \tag{1.2}$$

□

A linear map from $\mathfrak{g}$ to $\mathfrak{g}$ that satisfies eq.(1.2) is known as a *derivation*. The set of all derivations of $\mathfrak{g}$ forms a Lie algebra under commutator as bracket.

1.3 Derivations of a Lie Algebra

We continue from the previous Section but now choose a basis $\{e_1, e_2, ..., e_n\}$ for $\mathfrak{g}$ so that the dimension of $\mathfrak{g}$ is n. The structure constants C_{ij}^k of $\mathfrak{g}$ are defined by

$$[e_i, e_j] = C_{ij}^k e_k \tag{1.3}$$

where $1 \leq i < j \leq n$.

If we put $X = e_i$ and $Y = e_j$ in (1.2) we find that

$$D([e_i, e_j]) = [D(e_i), e_j] + [e_i, D(e_j)]$$

which gives

$$D(C_{ij}^k e_k) = [D_i^k e_k, e_j] + [e_i, D_j^k e_k]$$

and hence

$$D_k^m C_{ij}^k e_m = D_i^k C_{kj}^m e_m + D_j^k C_{ik}^m e_m$$

and so

$$D_k^m C_{ij}^k = D_i^k C_{kj}^m + D_j^k C_{ik}^m. \tag{1.4}$$

System (1.4) is a system of homogeneous linear equations for the entries D_j^i of the unknown matrix D. In principle there are $\frac{n^2(n-1)}{2}$ independent conditions in 1.4.

In our analysis of invariant Riemannian metrics we shall have occasion to study the space of derivations for each of the four-dimensional Lie algebras. Each such space can be readily computed by using MAPLE. However, we shall compute two examples "by hand" in order to get some idea of how the conditions work.

1.3.1 Derivations of Algebra 4.1

In our first example we consider the nilpotent Lie algebra 4.1: a complete list of the four-dimensional Lie algebras may be found below in Chapter 3 and the numbering is taken from [PSWZ]. The non-zero brackets are given by

$$[e_2, e_4] = e_1, [e_3, e_4] = e_2.$$

Accordingly, allowing for skew-symmetry the only non-zero structure constants are given by

$$C_{24}^1 = 1, C_{34}^2 = 1.$$

In principle there are 24 conditions but the non-zero conditions turn out to be

$$D_1^2 = D_1^3 = D_1^4 = D_2^4 = D_3^4 = D_1^1 - D_2^2 - D_4^4 = 0, D_1^2 - D_2^3 = 0, D_2^2 = D_3^3 + D_4^4. \quad (1.5)$$

The solution to (1.5) may be written as the 4×4 matrix

$$D = \begin{bmatrix} D_1^1 & D_2^1 & D_3^1 & D_4^1 \\ 0 & D_2^2 & D_2^1 & D_4^2 \\ 0 & 0 & 2D_2^2 - D_1^1 & D_4^3 \\ 0 & 0 & 0 & D_1^1 - D_2^2 \end{bmatrix}. \quad (1.6)$$

1.3.2 Derivations of Algebra 4.7

The non-zero brackets in 4.7 are given by $[e_1, e_4] = 2\,e_1, [e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$. Accordingly, allowing for skew-symmetry the only non-zero structure constants are given by

$$C_{14}^1 = 2, C_{23}^1 = 1, C_{24}^2 = 1, C_{34}^2 = 1, C_{34}^3 = 1$$

Again applying (1.4) with these structure constants, we get

$D^4_1 = -2D^4_4 = D^4_2 = D^3_1 = 0, D^3_1 - 2D^4_2 = -D^2_1 - 2D^4_3 = D^2_1 - D^3_1 = D^3_1 + D^4_2 = -D^1_2 - D^3_4 = -D^3_2 - D^4_4 = D^3_2 - D^4_4 = D^4_2 + D^4_3 = 0, D^1_1 - D^2_2 - D^3_3 = D^2_1 + D^4_2 - D^4_3 = -D^1_3 + D^1_2 + D^2_4 = D^2_2 - D^3_3 - D^4_4 = 0.$

$$ \tag{1.7}$$

The solution to (1.6) may be written as the 4×4 matrix

$$D = \begin{bmatrix} D^1_1 & D^1_2 & D^1_3 & D^1_4 \\ 0 & \frac{1}{2}D^1_1 & D^2_3 & D^1_3 - D^1_2 \\ 0 & 0 & \frac{1}{2}D^1_1 & -D^1_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1.4 Invariant metrics

A metric g on a Lie group G is said to be *invariant* if it is either left or right-invariant on G. Let us denote the Maurer-Cartan form (either left or right) on G by Ω. Thus

$$\Omega = [\omega_1, \omega_2, ..., \omega_n]. \tag{1.8}$$

Then g may be written as

$$\Omega g \Omega^t \tag{1.9}$$

where g is a *constant* matrix.

1.4.1 Heisenberg Lie Algebra

3.1: Non-zero brackets: $[e_2, e_3] = e_1$.

$$S = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case the right-invariant Maurer-Cartan form is given by $dz - ydx, dx, dy$. If we denote the matrix of the metric by $\left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & f \end{smallmatrix}\right]$ the metric is given by

$$a(dz - ydx)^2 + 2b(dz - ydx)dx + 2c(dz - ydx)dy + ddx^2 + 2edxdy + fdy^2. \tag{1.10}$$

We can compare (1.10) with a general three-dimensional Riemannian metric. In local terms it consists of a an arbitrary symmetric matrix

$$adz^2 + 2bdxdz + 2cdydz + ddx^2 + 2edxdy + fdy^2 \tag{1.11}$$

but now a, b, c, d, e, f are arbitrary *functions* of x, y, z. Despite the fact that an invariant metric is of a much restricted form, nonetheless it is still difficult to compute the curvature tensor. In this dissertation we propose a new technique; that is, we shall reduce the metric using the automorphism group of the algebra.

1.5 Unimodularity

Milnor [Mil] shows that in dimension three the Lie algebra bracket can be written in a special way; in fact it may be realized as the composition of the standard vector cross product on $\mathbb{R}^3$ followed by a certain linear transformation denoted by L. In the case where L is symmetric or self-adjoint, and only in that case, the Lie algebra is unimodular. A Lie algebra is said to be *unimodular* if every inner derivation has trace zero. Milnor [Mil] shows further in the unimodular case that each of the three brackets has just one term on the right hand side. Such a property does not hold in higher dimensions and we discuss the four-dimensional case at the beginning of Chapter 3.

1.6 Orthonormal Bases

In his paper on invariant metrics [Mil] Milnor uses throughout orthonormal bases. Let us investigate what the ramifications are of using such bases. In fact let us look at a particular three-dimensional Lie group as an example.

1.6.1 Example: Algebra 3.2

The corresponding matrix Lie group is given by

$$S = \begin{bmatrix} e^z & ze^z & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}.$$

The non-zero brackets are: $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$ and the left/right-invariant vector fields/one-forms are given by:

Left-invariant vector fields: $e^z D_x, e^z(D_y + zD_x), -D_z$

Left-invariant one forms: $e^{-z}(dx - zdy), e^{-z}dy, dz$

Right-invariant vector fields: $D_x, D_y, D_z + (x + y)D_x + yD_y$

Right-invariant one forms: $dx - (x + y)dz, dy - ydz, dz$.

The space of derivations is given by $\left[\begin{smallmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and the space of automorphisms of the Lie algebra by $\left[\begin{smallmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & 1 \end{smallmatrix}\right]$ where $a \neq 0$. We have chosen this example because it is an example of a non-unimodular Lie group and such groups were discussed in [Mil] pages 320ff. We shall denotes the inner product corresponding to the invariant metric by $\langle , \rangle$. We shall try now to construct an orthonormal basis by using Lie algebra automorphisms. First of all choose

$$\overline{e_3} = e_3 + ce_1 + de_2. \tag{1.12}$$

Then

$$\langle e_1, \overline{e_3}\rangle = \langle e_1, e_3\rangle + c\langle e_1, e_1\rangle + d\langle e_1, e_2\rangle. \tag{1.13}$$

so that if $\langle e_1, \overline{e_3}\rangle = 0$ then

$$c\langle e_1, e_1\rangle + d\langle e_1, e_2\rangle = -\langle e_1, e_3\rangle. \tag{1.14}$$

Similarly

$$c\langle e_1, e_2\rangle + d\langle e_2, e_2\rangle = -\langle e_2, e_3\rangle. \tag{1.15}$$

Now eq.(1.14, 1.15) can be solved for c and d since $\langle\, , \rangle$ is positive definite.

Now put

$$\overline{e_1} = ae_1, \overline{e_2} = be_1 + ae_2. \tag{1.16}$$

If $\{\overline{e_1}, \overline{e_2}\}$ are to be orthonormal then we must have that

$$a^2\langle e_1, e_1\rangle \quad = 1, \tag{1.17}$$

$$b\langle e_1, e_1\rangle + a\langle e_1, e_2\rangle \quad = 0 \tag{1.18}$$

$$b^2\langle e_1, e_1\rangle + 2ab\langle e_1, e_2\rangle + a^2\langle e_2, e_2\rangle \quad = 1. \tag{1.19}$$

Eqs.(1.17,1.18,1.19) comprise three conditions on the two unknowns a, b so it is not possible in general to find a solution. The conclusion is that it is not possible to start with a Lie algebra in a specific basis and to simultaneously put the metric into orthonormal form. Either one starts with the Lie algebra in a specific form and uses a non-orthonormal basis or, if one uses an orthonormal basis, the form of the Lie algebra will become more complicated involving extra terms and coefficients in the brackets.

In [Mil] Milnor first of all considers unimodular Lie algebras in dimension three. He then moves onto the non-unimodular case. He shows first of all that there exists an orthonormal basis with respect to which the Lie brackets may be written as

$$[e_1, e_3] = \alpha e_1 + \beta e_2, [e_2, e_3] = \gamma e_1 + \delta e_2. \tag{1.20}$$

It will be convenient for what comes later to consider the more general situation where we have

$$[e_i, e_n] = C_i^j e_j \tag{1.21}$$

and $1 \leq i, j, k, m \leq n-1$ as the only non-zero brackets of an n dimensional Lie algebra. Let us make a change of basis of the form

$$\overline{e_i} = P_i^k e_k, \overline{e_n} = e_n. \tag{1.22}$$

The brackets in the new basis are given by

$$[\overline{e_i}, \overline{e_n}] = \overline{C}_i^m \overline{e_m} \tag{1.23}$$

where

$$\overline{C}_i^m = {P^{-1}}_j^m C_k^j P_i^k \tag{1.24}$$

or in matrix notation

$$\overline{C} = P^{-1} C P \tag{1.25}$$

so that C is acted on by conjugation.

Let us come back to the case $n = 2$ for which $C = \left[\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right]$. We shall choose P to be the rotation $\left[\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right]$ in the (e_1, e_2)-plane. As such we find that

$$\overline{C} = \left[\begin{smallmatrix}\overline{\alpha} & \overline{\gamma}\\ \overline{\beta} & \overline{\delta}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\alpha\cos^2\theta-(\beta+\gamma)\cos\theta\sin\theta+\delta\sin^2\theta & \gamma\cos^2\theta+(\alpha-\delta)\cos\theta\sin\theta-\beta\sin^2\theta\\ \beta\cos^2\theta+(\alpha-\delta)\cos\theta\sin\theta-\gamma\sin^2\theta & \delta\cos^2\theta+(\beta+\gamma)\cos\theta\sin\theta+\alpha\sin^2\theta\end{smallmatrix}\right]. \tag{1.26}$$

Now we choose θ such that $\overline{\alpha}\,\overline{\gamma} + \overline{\beta}\,\overline{\delta} = 0$ which gives

$$2(\alpha\gamma + \beta\delta)\cos 2\theta + (\alpha^2 + \beta^2 - \gamma^2 - \delta^2)\sin 2\theta = 0. \tag{1.27}$$

It is clearly possible to chose θ so that eq.(1.27) is satisfied. We shall continue this discussion in the next Chapter after we have introduced some ways to find curvature.

1.7 Origin of the Einstein Condition

The interest in Einstein manifolds was motivated originally by the theory of relativity. According to Einstein's general theory of relativity,

$$G_{ij} = 8\pi T_{ij}. \tag{1.28}$$

Equations (1.28) are known as *Einstein's field equations*, where T_{ij} is the *stress-energy tensor*, $G_{ij} = R_{ij} - kg_{ij}$ is the *Einstein tensor*, with R_{ij} the Ricci curvature tensor and $k = \Lambda - \frac{\rho}{2}$; ρ, the scalar curvature and Λ, the cosmological constant.

The unknowns in Equations (1.28) are the components of the metric g_{ij}: in general relativity the "metric" is a four-dimensional Lorentzian metric. The Ricci curvature in a system of local coordinates (x^i) is given by

$$R_{kl} = g^{im}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g^{im} g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}) \tag{1.29}$$

where the connection components or Christoffel symbols are given by

$$\Gamma^n_{kl} = \frac{1}{2} g^{mn} \left(\frac{\partial g_{km}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{1.30}$$

Equations (1.28) comprise a very complicated system of second order non-linear partial differential equations.

1.7.1 Einstein Space

In a vacuum (a region of space-time with no matter) $T_{ij} = 0$ and one can rewrite Einstein's equation in the form

$$R_{ij} = k g_{ij}. \tag{1.31}$$

Such a space is known as an *Einstein space*. Einstein spaces with $k = 0$ are called *Ricci-flat* manifolds. Not many examples of Einstein manifolds are known but we do of course refer to the encyclopaedic reference [Be]. Mathematicians are primarily interested in compact Einstein spaces. For a recent survey we refer the reader to [An].

Chapter 2

Curvature Formulas

2.1 Formulas for curvature and Ricci tensors

In this Section we derive a formula for the Ricci tensor of a left or right-invariant metric g_{ij} on a Lie group G. It is convenient to keep the metric visible since we cannot assume that we are working in an orthonormal frame. Thus if g_{ij} is right-invariant and $\{e_i\}$ is a basis for the right-invariant vector fields, g_{ij} is a constant matrix. The compatibility of g_{ij} with its Levi-Civita connection Γ^i_{jk} gives

$$g_{lj}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj} = 0 \tag{2.1}$$

whereas the fact that the connection is torsion-free gives

$$\Gamma^l_{ki} - \Gamma^l_{ik} = C^l_{ki}. \tag{2.2}$$

From $\nabla_{e_k} e_i - \nabla_{e_i} e_k = [e_k, e_i]$, we have $\Gamma^l_{ki} e_l - \Gamma^l_{ik} e_l = C^l_{ki} e_l$. Now cycle the indices in eqn.(2.1) and use eqn.(2.2) to obtain

$$\Gamma^m_{ij} = \frac{1}{2} g^{km} (g_{il} C^l_{kj} + g_{jl} C^l_{ki} + g_{kl} C^l_{ij}). \tag{2.3}$$

Notice that in eqn.(2.3) the first two terms are symmetric in i and j and that the third term is skew-symmetric in i and j. Now we find that the curvature is given by

$$R^i_{ljk} = \Gamma^m_{jl}\Gamma^i_{km} - \Gamma^m_{kl}\Gamma^i_{jm} + \Gamma^i_{ml}C^m_{jk}. \tag{2.4}$$

Notice that if we write eqn.(2.4) invariantly we get the usual formula for curvature, that is,

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \tag{2.5}$$

where in the last term we have the Lie algebra bracket. After a very long calculation eqn.(2.4) gives for the Ricci tensor

$$\begin{aligned} 4R_{ij} = {} & 2g^{kp}g_{mq}C^m_{jk}C^q_{pi} - g^{kq}g^{mn}g_{ip}g_{jr}C^p_{nk}C^r_{qm} \\ & -2C^k_{mk}g^{mp}(g_{jq}C^q_{pi} + g_{iq}C^q_{pj}) + 2C^m_{jk}C^k_{mi}. \end{aligned} \tag{2.6}$$

We note that R_{ij} is symmetric as it must be for a metric connection and also that the fourth term is the Killing form up to a factor. Moreover the third term vanishes for a unimodular Lie algebra and the third and fourth terms vanish for a nilpotent Lie algebra.

If we assume that we are in an orthonormal frame so that $g_{ij} = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, then we find after very long calculations, that the curvature and Ricci tensors and scalar curvature are given by:

$$\begin{aligned} 4R^i_{ljk} = {} & \sum_m C^m_{jk}(C^l_{im} - C^i_{lm}) + C^m_{il}(C^k_{jm} - C^j_{km}) + C^m_{jl}C^m_{ik} - C^m_{kl}C^m_{ij} \\ & +2C^m_{jk}C^m_{il} + (C^j_{ml} + C^l_{mj})(C^k_{im} + C^i_{km}) - (C^k_{ml} + C^l_{mk})(C^j_{im} + C^i_{jm}). \end{aligned} \tag{2.7}$$

$$4R_{lj} = \sum_{k,m} 2(C^j_{ml} + C^l_{mj})C^k_{km} + 2C^m_{jk}C^m_{kl} - 2C^m_{jk}C^k_{lm} + C^j_{km}C^l_{km}. \tag{2.8}$$

$$4\rho = -(4\sum_m(\sum_j C^j_{mj})^2 + \sum_{j,k,m}((C^k_{mj})^2 + 2C^m_{jk}C^k_{jm})). \tag{2.9}$$

2.1.1 Codimension one abelian nilradical

Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then the non-zero brackets are given by

$$[e_i, e_n] = C^j_{in} e_j. \tag{2.10}$$

For the moment we shall suppress the index n and work with the $(n-1) \times (n-1)$ matrix C whose (j,i)th entry $C^j_i = C^j_{in}$. Thus, all the information about the structure constants is embodied in the matrix C where the last row and column of C are zero.

Proposition 2.1.1. *Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then each of the matrices E_i for $1 \leq i \leq n-1$, where the only non-zero entry of E_i is 1 in the (i,n)th position, is a derivation of $\mathfrak{g}$.*

Proof. A derivation D has to satisfy the two conditions $1 \leq i < j \leq n-1$

$$D[e_i, e_j] = [De_i, e_j] + [e_i, De_j],\ D[e_i, e_n] = [De_i, e_n] + [e_i, De_n].$$

For $D = E_k$ $1 \leq k \leq n-1$, the first condition is immediate because $[e_i, e_j] = 0$ and e_i, e_j are in the null space of E_k. For the second condition $[e_i, e_n]$ and e_i are in the null space of E_k whereas $E_k(e_n) = e_k$ and $[e_i, e_k] = 0$. □

Corollary 2.1.1. *Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian nilradical spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then each of the matrices E_i for $1 \leq i \leq n-1$,* *where the only non-zero entry of E_i is a 1 in the (i,n)th position, is a derivation of $\mathfrak{g}$.*

Proposition 2.1.2. *Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$ and has trivial center. Then if D is a derivation of $\mathfrak{g}$ we have $D_k^n = 0$ where $1 \leq k \leq n-1$. Furthermore, if $\mathfrak{g}$ is not unimodular then also $D_n^n = 0$.*

Proof. Note first of all that $Z = Z^i e_i + Z^n e_n$ is in the center of $\mathfrak{g}$ if and only if $Z^n C_j^k = 0$ and $Z^i C_i^j = 0$. The first of these conditions implies that $Z^n = 0$. The second condition has a non-zero solution if an only the matrix is singular.

Next a derivation D satisfies

$$D[e_i, e_n] = [De_i, e_n] + [e_i, De_n].$$

Now $D[e_i, e_n] = D(C_i^j e_j) = C_i^j (D_j^k e_k + D_j^n e_n)$. On the other hand

$$\begin{aligned}[De_i, e_n] + [e_i, De_n] &= [D_i^j e_j + D_i^n e_n, e_n] + [e_i, D_n^j e_j + D_n^n e_n] \\ &= D_i^j C_j^k e_k + D_n^n C_i^k e_k.\end{aligned}$$

If we equate the coefficients of e_n we obtain

$$C_i^j D_j^n = 0.$$

Hence if C as an $(n-1) \times (n-1)$ is non-singular then $D_j^n = 0$.

On the other hand if we equate the coefficients of e_k we obtain

$$C_i^j D_j^k = D_i^j C_j^k + D_n^n C_i^k$$

which may be written as

$$tr([DD, C]) = D^n_n tr(C)$$

where DD denotes the $(n-1)\times(n-1)$ submatrix of D consisting of the first $(n-1)$ rows and first $(n-1)$ columns. However, the last equation gives $D^n_n tr(C) = 0$ since the trace of a commutator is zero. The only ad-matrix that is not nilpotent and hence may not have trace zero is ad $(e_n) = -C$. Hence if $\mathfrak{g}$ is not unimodular then $D^n_n = 0$. □

In particular it follows that a Lie algebra that has a codimension one abelian ideal and is not unimodular has no non-singular derivations. In fact Jacobson [Jac] showed that if a Lie algebra that has a non-singular derivation then it must be nilpotent.

To continue we refer to (2.10) and we shall restore the subscript n. We write our Lie algebra $\mathfrak{g}$ as

$$[e_i, e_j] = C^k_{ij} e_k, [e_i, e_n] = C^k_{in} e_k. \tag{2.11}$$

For fixed i the matrix C^k_{ij} is the same as ad(e_i). To say that $\mathfrak{g}$ has a codimension one abelian nilradical spanned by $\{e_1, e_2, ..., e_{n-1}\}$ is the same as saying that the matrices C^k_{ij} for fixed $i(1 \leq i \leq n-1)$, are zero. Furthermore the matrix $C^k_{nj} = -C^k_{jn}$ has nth column zero because of skew-symmetry and nth row zero because of the ideal condition.

Now we can apply Proposition 2.1.2 to the reduction of an invariant metric on the Lie group G whose Lie algebra $\mathfrak{g}$ has a codimension one abelian nilradical. Thus we may assume that the metric g has zeroes in the (i, n)th and (n, i)th entries for $1 \leq i \leq n-1$. We note that the inverse of g will also be of the same form. We use formula (2.6):

$$\begin{aligned} 4R_{ij} = {} & 2g^{kp} g_{mq} C^m_{jk} C^q_{pi} - g^{kq} g^{ms} g_{ip} g_{jr} C^p_{sk} C^r_{qm} \\ & -2C^k_{mk} g^{mp}(g_{jq} C^q_{pi} + g_{iq} C^q_{pj}) + 2C^m_{jk} C^k_{mi}. \end{aligned} \tag{2.12}$$

In what follows the summation convention on repeated indices remains in effect except with regard to the index n. Now assume that $i < n$ and $j = n$. As regards the first term $g^{kp} g_{mq} C^m_{jk} C^q_{pi}$ we have $j = n$ and we may assume $k < n$ and $p = n$ so $g^{kp} = g^{kn} = 0$. For the second term since $j = n$, we may assume that $r < n$ which gives $g_{jr} = g_{nr} = 0$. Concerning the third term and discounting the factor of -2 we have $C^k_{mk} g^{mp}(g_{jq} C^q_{pi} + g_{iq} C^q_{pj})$. Taking the first of these terms as $j = n$, we may assume that $q < n$ so that $g_{nq} = 0$ and $C^k_{mk} g^{mp} g_{nq} C^q_{pi} = 0$. In the second term we may assume that $m = n$ (otherwise the trace of $C^k_{mk} = 0$) and hence $p = n$ giving $C^k_{mk} g^{mp} g_{iq} C^q_{pn} = 0$.

Finally the last term for $i < n, j = n$, we may assume $m = n$ so that $2C^m_{nk} C^k_{mi} = 2C^n_{nk} C^k_{ni} = 0$. Hence:

Proposition 2.1.3. *Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then after reducing g so that $g_{in} = g_{ni} = 0$ where $i < n$ then $R_{in} = R_{ni} = 0$ where R_{ij} denotes the Ricci tensor as a type $(0,2)$-tensor. Furthermore the Ricci tensor as a type $(1,1)$-tensor enjoys the same property despite not being necessarily symmetric and $\frac{R_{nn}}{g_{nn}}$ is an eigenvalue.*

Now we come back to (2.10). We note that if we use a change of basis matrix P in the ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$ and keep e_n fixed then the matrix C changes by $P^{-1}CP$. As such we may choose P and scale e_n so as to reduce to an orthonormal basis. Then, (2.8) gives

$$4R_{nn} = \sum_{k,m} -2(C^m_{nk} C^m_{nk} + C^m_{nk} C^k_{nm}) \tag{2.13}$$

which may be written as

$$4R_{nn} = -2(||C||^2 + tr(C^2)). \tag{2.14}$$

where $||C||$ denotes the Frobenius or Hilbert-Schmidt norm of a matrix. Again (2.14) may be rewritten as

$$4R_{nn} = -2(2\sum_k (C_{nk}^k)^2 + \sum_{k\neq m}(C_{nm}^k + C_{nk}^m)^2). \tag{2.15}$$

Thus $R_{nn} \leq 0$ and $R_{nn} = 0$ if and only if C is skew-symmetric in the orthonormal basis.

Let us now consider (2.9) for the case where the Lie algebra is given by (2.10). Milnor [Mil] has shown that whenever the Lie algebra is solvable that $\rho \leq 0$. In the case at hand we find

$$4\rho = -(4(\sum_j C_{nj}^j)^2 + \sum_{j,k,m}((C_{mj}^k)^2 + 2C_{jk}^m C_{jm}^k)) \tag{2.16}$$

which may be rewritten as

$$4\rho = -(4(\sum_j C_{nj}^j)^2 + \sum_m ||C_m||^2 + 2C_{jk}^m C_{jm}^k)) \tag{2.17}$$

where $C_m = \mathrm{ad}(e_m)$ for $1 \leq m \leq n-1$ and $C_n = C$. However, $\sum_m ||C_m||^2 = 2||C||^2$ and so

$$4\rho = -(4(tr(C))^2 + 2||C||^2 + 2tr(C^2)). \tag{2.18}$$

Comparing (2.14) and (2.18) we have that

$$\rho - R_{nn} = -(tr(C))^2. \tag{2.19}$$

Now we know that R_{nn} is one Ricci eigenvalue and that $\rho - R_{nn}$ is the trace of the submatrix of Ricci consisting of the first $n-1$ rows and first $n-1$ columns. Since $\rho - R_{nn} \leq 0$ it follows that Ricci has at least two eigenvalues that are not positive.

Proposition 2.1.4. *Suppose that the Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then Ricci has at least two eigenvalues that are not positive.*

Corollary 2.1.2. *Suppose that the Lie algebra $\mathfrak{g}$ is nilpotent and has a codimension one abelian ideal spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Then if $\mathfrak{g}$ is not abelian, Ricci will have at least two not positive eigenvalues and one eigenvalue is negative the sum of $n-2$ others (ignoring multiplicity).*

Proof. Observe first of all that from (2.19) that the matrix C must be nilpotent if $\mathfrak{g}$ is nilpotent by Engel's theorem. Hence $\rho = R_{nn}$ and the sum of the first $n-1$ eigenvalues is zero. $\square$

2.1.2 Einstein spaces

Now we shall take up the question of whether an invariant metric on a Lie group whose Lie algebra has a codimension one abelian nilradical can correspond to an Einstein space. We start from eq.(2.8). We note first of all that in the term $\sum_k 2(C^j_{ml} + C^l_{mj})C^k_{km}$ we may assume $m = n$ (without sum) since for fixed m we have $C^i_{mj} = \mathrm{ad}(e_m)$ for $1 \leq m \leq n-1$ is nilpotent because of Engel's Theorem. As regards $\sum_k 2C^m_{jk}C^m_{kl}$ we may assume again that $k = n$ (without sum) and that $m < n$. In the term $\sum_k -2C^m_{jk}C^k_{lm}$ we have that $k, m < n$ and hence $j = l = n$. In the term $\sum_{k,m} C^j_{km}C^l_{km}$ we may assume that $k = n$ or $m = n$ and so we have that $\sum_{k,m} C^j_{km}C^l_{km} = 2\sum_m C^j_{nm}C^l_{nm}$. Since C has zero last row and column we can express the $(n-1)\times(n-1)$ submatrix of R_{lj} by deleting the last row and column as

$$Ricci = 2(-(C + C^t)tr(C) - C^tC + CC^t). \qquad (2.20)$$

We have been assuming in this subsection that the $\{e_1, e_2, ..., e_n\}$ form an orthonormal basis. As such we have the freedom to make an orthogonal change of basis. In particular we can make an orthogonal change in the subspace spanned by $\{e_1, e_2, ..., e_{n-1}\}$. Now in eq.(2.20) the matrix C may be written as

$$C = A + S. \tag{2.21}$$

where A is symmetric and S is skew-symmetric. If P is orthogonal then

$$P^t C P = P^t A P + P^t S P \tag{2.22}$$

and P may be chosen so as to make $P^t A P$ diagonal. As such $P^t S P$ will remain skew-symmetric. In other words, without loss of generality we may assume in eq. (2.21) that A is diagonal.

Now using eq.(2.21) in eq.(2.20) we find first of all that $(C + C^t)tr(C) = 2tr(A)A$ and

$$\begin{aligned} & (A+S)(A^t+S^t) - (A^t+S^t)(A+S) \\ = \; & A^2 - AS + SA - S^2 - A^2 - AS + SA + S^2 \\ = \; & 2(SA - AS). \end{aligned}$$

We remark also that $SA - AS$ is symmetric. Thus

$$Ricci = 2(-tr(A)A + SA - AS). \tag{2.23}$$

Let us put $A = diag(a_1, a_2, ..., a_n)$ and let the (i, j)th entry of S be s_{ij}. Then the (i, j)th entry of $SA - AS$ is given by $(a_i - a_j)s_{ij}$ where there is no sum over i or j: in particular the diagonal entries of $SA - AS$ are zero. Suppose now that we want Ricci to be a multiple of the identity so that we have an Einstein space. Then with reference to (2.23) we distinguish between the cases $tr(A) \neq 0$ and $tr(A) = 0$. In the former case we obtain immediately that A is a multiple of the identity. We do indeed obtain an Einstein space but it has been shown recently [Tho] that such a space is of constant sectional curvature. The first example of this phenomenon was noted by Milnor [Mil].

In second case where $tr(A) = 0$ since $SA - AS$ has zero diagonal entries if there is an Einstein space it must have zero Ricci tensor, that is, it is Ricci-flat. In particular $R_{nn} = 0$ but we have seen from eq.(2.15) and the remark immediately following it, that in this case the matrix C must be skew-symmetric. Again this situation is discussed in [Tho]. Such a space is not just Ricci-flat but actually flat: the Lie algebra can be decomposed into a part on which is C non-singular skew-symmetric and an abelian part. The simplest case where is C non-singular comes from Algebra 3.6 in [PSWZ], which is the Lie algebra of the Euclidean group of plane and was mentioned in [Mil].

Finally, as regards a Lie algebra that is nilpotent and not abelian, Milnor has shown that there are always directions of positive and negative Ricci curvature and so it never corresponds to an Einstein space.

Theorem 2.1.1. *Suppose a Lie algebra $\mathfrak{g}$ has a codimension one abelian ideal. Then if $\mathfrak{g}$ is nilpotent it does not correspond to an Einstein space. In the solvable not nilpotent case $\mathfrak{g}$ is associated to an Einstein space if and only if it is a space of constant sectional curvature.*

In the case where there is a space of *non-zero* constant sectional curvature the Lie algebra is of odd dimension and is such that

$$C = aI + S. \tag{2.24}$$

where S is a non-singular skew-symmetric matrix. In fact, although it has not been proved here, such spaces are precisely the ones which are associated to invariant metrics of constant negative sectional curvature on a Lie group: in other words the hypothesis that the Lie algebra should be solvable with a codimension one abelian nilradical is necessary as well as being sufficient [Tho]. In the case where the space is of constant *zero* curvature the condition of having a codimension one abelian nilradical is not necessary; however, the Lie algebra splits as a direct sum of an abelian algebra and a solvable algebra that has an abelian nilradical and an abelian complement. In that case there are several commuting skew-symmetric matrices. For further details the reader is referred to [Mil] and [Tho].

2.2 Products

Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ where $\mathfrak{k}$ is an abelian Lie algebra. We use a basis $\{e_a\}, (1 \leq a \leq p)$ for $\mathfrak{h}$ and a basis $\{e_i\}, (p+1 \leq i \leq q)$ for $\mathfrak{k}$. Suppose that $[e_a, e_b] = C^c_{ab} e_c$. Now suppose that D is a derivation of $\mathfrak{g}$ so that

$$D(e_a) = \lambda^b_a e_b + \mu^i_a e_i,\ D(e_i) = \rho^a_i b e_a + \sigma^j_i e_j.$$

The conditions for D to be a derivation are

$$\rho^b_i C^c_{ab} = 0,\ C^c_{ab} \mu^i_c = 0$$

and λ^b_a defines a derivation of $\mathfrak{h}$ and σ^j_i is arbitrary. Now D is given by the matrix $\begin{bmatrix} \lambda^a_b & \rho^a_j \\ \mu^i_b & \sigma^i_j \end{bmatrix}$.

If $Z(\mathfrak{h}) = 0$ then $\rho^b_i = 0$ and the condition on the μ^i_c's give non-trivial solutions if and only if $[\mathfrak{h}, \mathfrak{h}] \neq \mathfrak{h}$. Let us assume that $\mathfrak{h}$ is solvable and that $Z(\mathfrak{h}) = 0$ and that $e_{m+1}, ..., e_p$ span a complement to $[\mathfrak{h}, \mathfrak{h}]$. Then the matrix μ is of the form $\left[0_{q\times m} | M_{q\times(p-m)} \right]$ where M is arbitrary. Hence:

Proposition 2.2.1. *Suppose that the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ where $\mathfrak{k}$ is an abelian Lie algebra. Let a basis $\{e_a\}, (1 \le a \le p)$ be chosen for $\mathfrak{h}$ and $\{e_i\}, (p+1 \le i \le q)$ for $\mathfrak{k}$. Suppose further that this basis is chosen so that $\{e_1, e_2, ..., e_m\}$ spans $[\mathfrak{h}, \mathfrak{h}]$ and that $Z(\mathfrak{h}) = 0$. Then the space of derivations is given by $\begin{bmatrix} \lambda^a_b & 0 \\ \mu^i_b & \sigma^i_j \end{bmatrix}$ where λ is a derivation of $\mathfrak{h}$, the matrix μ is of the form $\left[0_{q\times m} | M_{q\times(p-m)} \right]$ where M and σ are arbitrary.*

Now we use the formula for Ricci and we find that

$$\begin{aligned} 4R_{ab} &= 2g^{fh}g_{ed}C^e_{bf}C^d_{ha} - g^{fh}g^{ge}g_{ac}g_{bd}C^c_{ef}C^d_{hg} \\ &\quad -2C^f_{ef}g^{ed}(g_{bc}C^c_{da} + g_{ac}C^c_{db}) + 2C^c_{bd}C^d_{ca}. \end{aligned} \tag{2.25}$$

$$4R_{aj} = -g^{df}g^{hb}g_{ac}g_{je}C^c_{bd}C^e_{fh} - 2C^c_{bc}g^{be}g_{jd}C^d_{ea} \tag{2.26}$$

$$4R_{ij} = -g^{ab}g^{fe}g_{ic}g_{jd}C^c_{ea}C^d_{bf}. \tag{2.27}$$

We shall use formulas (2.25, 2.26, 2.27) in Section 3.14. Formally (2.25) looks like the Ricci tensor for g_{ab}; however, g^{ab} also depends on g_{aj} and g_{ij}.

2.3 Continuation of the three-dimensional case

Now we shall resume from Section 1.6 and in particular eq.(1.20). We find that the Ricci tensor is given by

$$R = \frac{1}{2}\begin{bmatrix} \gamma^2 - \beta^2 - 2\delta\,\alpha - 2\alpha^2 & -2(\alpha\,\gamma + \delta\,\beta) & 0 \\ -2(\alpha\,\gamma + \delta\,\beta) & -\gamma^2 + \beta^2 - 2\delta\,\alpha - 2\delta^2 & 0 \\ 0 & 0 & -2\alpha^2 - \gamma^2 - \beta^2 - 2\gamma\,\beta - 2\delta^2 \end{bmatrix}$$

in agreement with [Mil]. According to Section 1.6 we argued that it is possible to choose the orthonormal basis so that $\alpha\gamma + \delta\beta = 0$ in which case the Ricci tensor is diagonal again in agreement with [Mil]. However, it will not be possible to carry out a similar analysis even in dimension four.

Chapter 3

Four Dimensional Lie Algebras

In this Chapter we begin our systematic study of the four-dimensional Lie groups. We begin with a summary of the structure of indecomposable, four-dimensional Lie algebras. It is convenient for us to consider that there are ten classes of such algebra in dimension four. In succeeding Sections we discuss each algebra in turn and draw various conclusions about the curvature properties of an invariant metric on an associated Lie group. We shall devote a separate Section to each of the ten class of algebras. We shall in each case give a list of the non-zero brackets, a group matrix whose Lie algebra engenders the given algebra and the left/right-invariant vector fields/ one forms.

3.1 Four Dimensional Lie Algebras

As regards the four-dimensional algebras we consider first of all the indecomposable algebras which means that the algebra is not a direct sum of lower-dimensional algebras. The *real* indecomposable, four-dimensional Lie algebras were classified by G. Mubarakzyanov: see [Mub] and references therein. They can be found easily in [PSWZ]. Following [PSWZ] there are twelve classes of such algebra in dimension four: the first is nilpotent and the remaining eleven are solvable and in particular none are semi-simple. The classification of solvable algebras necessarily involves some essential parameters and therefore there is always a certain amount of arbitrariness in the classification. In [PSWZ], the Algebras 4.8 and 4.10 are simply the limiting cases of Algebras 4.9b and 4.11a for which $b = -1$ and $a = 0$, respectively. Therefore we shall assume that there are actually ten classes of algebra. Of these ten, two Algebras - 4.5ab and 4.6ab - depend on two parameters. Algebras 4.2a, 4.8/9b, 4.10/11a depend on a single parameter whereas Algebras 4.1, 4.3, 4.4, 4.7 and 4.12 do not depend on a parameter at all.

We remark that the first algebra 4.1 is nilpotent and the next five $4.2 - 4.6$ have a three-dimensional abelian nilradical; these algebras are classified, up to isomorphism, by the projective class of the Jordan normal form of $\mathrm{ad}(e_4)$ where e_4 spans a complement to the nilradical. Algebras $4.7 - 4.11$ have the three-dimensional Heisenberg algebra as their nilradical and 4.12 has a two-dimensional abelian nilradical.

As regards algebras $4.7 - 4.11$ above they can be obtained by noting that there is a one-one correspondence between three-dimensional algebras that have an abelian nilradical and four-dimensional algebras that have the Heisenberg algebra as nilradical. Unfortunately such a simple correspondence does not exist in higher dimensions.

Finally Algebra 4.12 is decomposable over $\mathbb{C}$. If we make the change of basis

$$\overline{e_1} = \frac{1}{2}(e_1 + ie_2), \overline{e_2} = \frac{1}{2}(e_1 - ie_2), \overline{e_3} = \frac{1}{2}(e_3 + ie_4), \overline{e_4} = \frac{1}{2}(e_3 - ie_4) \tag{3.1}$$

the algebra becomes

$$[\overline{e_1}, \overline{e_4}] = \overline{e_1},\ [\overline{e_2}, \overline{e_3}] = \overline{e_3}. \tag{3.2}$$

The following four-dimensional indecomposable algebras are precisely the unimodular ones: Algebras $4.1, 4.2(p = -2), 4.5(a + b + 1 = 0), 4.6(a + 2b = 0), 4.8, 4.10$. Of these algebras, the brackets in the Algebras $4.2(p = -2), 4.6$ do not have a just single term on the right hand side.

3.2 Invariant metric

The matrix of a general four-dimensional left-invariant metric, relative to the Lie algebra basis, is given by the symmetric matrix

$$g = \begin{bmatrix} a & b & c & d \\ b & e & f & k \\ c & f & h & i \\ d & k & i & j \end{bmatrix}. \tag{3.3}$$

The matrix g is positive definite because the Riemannian metric is positive definite.

3.3 Algebra 4.1

4.1 $[e_2, e_4] = e_1, [e_3, e_4] = e_2$

$$S = \begin{bmatrix} 1 & w & \frac{w^2}{2} & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields: $D_x, D_y + wD_x, D_z + wD_y + \frac{w^2}{2} D_z, D_w$

Left-invariant one forms: $dx - wdy + \frac{1}{2}w^2 dz, dy - wdz, dz, dw$

Right-invariant vector fields: $D_x, D_y, D_z, D_w + yD_x + zD_y$

Right-invariant one-forms $dx - ydw, dy - zdw, dz, dw$.

3.3.1 Reduction of metric

The space of derivations of Algebra 4.1 is seven-dimensional and is given by the following matrix as we saw in eq.(1.6)

$$der(g) = \begin{bmatrix} \phi & \alpha & \beta & \gamma \\ 0 & \theta & \alpha & \delta \\ 0 & 0 & 2\theta - \phi & \epsilon \\ 0 & 0 & 0 & \phi - \theta \end{bmatrix}. \tag{3.4}$$

Now we proceed to reduce the metric g in eq.(3.3) using the entries in (3.4) one at a time in the following way. In (2.1) put $\alpha = 1$ and the remaining entries zero to obtain a matrix M; then put $P = e^{tM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} 1 & t & \frac{t^2}{2} & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

Now P acts on the metric g to give g_t by $P^T g P$ so that

$$g_t = \begin{bmatrix} a & ta+b & \frac{at^2}{2}+bt+c & d \\ ta+b & at^2+2bt+e & \frac{at^3}{2}+\frac{3bt^2}{2}+(c+e)t+f & td+k \\ \frac{at^2}{2}+bt+c & \frac{at^3}{2}+\frac{3bt^2}{2}+(c+e)t+f & \frac{at^4}{4}+bt^3+(c+e)t^2+2ft+h & \frac{dt^2}{2}+kt+i \\ d & td+k & \frac{dt^2}{2}+tk+i & j \end{bmatrix}. \tag{3.6}$$

Looking at (3.6) since, $a > 0$, choose $t = -\frac{b}{a}$. What we have shown, is that, without loss of generality, it may be assumed $b = 0$ in eq.(3.4.1).

Next in 3.4, put $\beta = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M put $P = e^{sM}$. Then P is given by

$$P = e^{sM} = \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

so that

$$g_s = P^T g P = \begin{bmatrix} a & 0 & as+c & d \\ 0 & e & f & k \\ as+c & f & as^2+2cs+h & ds+i \\ d & k & ds+i & j \end{bmatrix}. \tag{3.8}$$

Again, since $a \neq 0$, we may choose, $s = -\frac{c}{a}$ in g_s so that $c = 0$ in the matrix g.

Next in 3.4, put $\gamma = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M find $P = e^{rM}$. Then P is given by

$$P = e^{rM} = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

so that

$$g_r = P^T g P = \begin{bmatrix} a & 0 & 0 & ra+d \\ 0 & e & f & k \\ 0 & f & h & i \\ ra+d & k & i & ar^2+2dr+j \end{bmatrix}. \tag{3.10}$$

Again, since $a \neq 0$, choose $r = -\frac{d}{a}$ in g_r so that $d = 0$ in the matrix g.

Next in 3.4, put $\delta = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M, find $P = e^{pM}$. Then P is given by

$$P = e^{pM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

so that

$$g_p = P^T g P = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & f & pe+k \\ 0 & f & h & pf+i \\ 0 & pe+k & pf+i & ep^2+2kp+j \end{bmatrix}. \tag{3.12}$$

Again, since, $e \neq 0$, choose,$p = -\frac{g}{e}$ in G_p so that $k = 0$ in the matrix g.

Next in 1.2, put $\epsilon = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M find $P = e^{qM}$. Then P is given by

$$P = e^{qM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.13}$$

so that

$$g_q = P^T g P = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & f & fq \\ 0 & f & h & hq+i \\ 0 & fq & hq+i & hq^2+2iq+j \end{bmatrix}. \tag{3.14}$$

Again, since $h \neq 0$, choose $q = -\frac{i}{h}$ in g_q so that $i = 0$ in the matrix g.

At this point having used $\alpha, \beta, \gamma, \delta, \epsilon$ we only have a diagonal matrix of derivations left to simplify g which is now of the form

$$g = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & f & 0 \\ 0 & f & h & 0 \\ 0 & 0 & 0 & j \end{bmatrix}. \tag{3.15}$$

In 3.4, put $\phi = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M, find $P = e^{uM}$. Then P is given by

$$P = e^{uM} = \begin{bmatrix} \mathrm{e}^{u} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-u} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{u} \end{bmatrix}. \tag{3.16}$$

Next in 3.4, put $\theta = 1$ and set all other Greek letters to zero. Again, denoting the resulting matrix by M put $P = e^{vM}$. Then P is given by

$$P = e^{vM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{v} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2v} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-v} \end{bmatrix}. \tag{3.17}$$

Finally we consider an overall scaling by e^{w} so that we consider a change of basis of the form

$$P = \begin{bmatrix} \mathrm{e}^{u+w} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{v+w} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2v-u+w} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{u-v+w} \end{bmatrix} \tag{3.18}$$

so that

$$\overline{g} = P^T g P = \begin{bmatrix} \mathrm{e}^{2(u+w)}a & 0 & 0 & 0 \\ 0 & \mathrm{e}^{2(v+w)}e & \mathrm{e}^{3v-u+2w}f & 0 \\ 0 & \mathrm{e}^{3v-u+2w}f & \mathrm{e}^{2(2v-u+w)}h & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2(u-v+w)}j \end{bmatrix}. \tag{3.19}$$

Now we choose u, v, w such that

$$\mathrm{e}^{2(u+w)}a = 1, \mathrm{e}^{2(v+w)}e = 1, \mathrm{e}^{2(u-v+w)}j = 1 \tag{3.20}$$

which is possible since each of $a, e, j > 0$.

Thus the metric g has been reduced to

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & f & 0 \\ 0 & f & h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.21}$$

where $det(g) = h - f^2 > 0$.

3.3.2 Ricci tensor

Now we shall investigate some curvature properties. First of all we find that the Ricci tensor in type $(0,2)$-form is given by

$$Ricci(0,2) = \frac{1}{2(h-f^2)} \begin{bmatrix} h & -f & -f^2 & 0 \\ -f & f^2-h+1 & f & 0 \\ -f^2 & f & 2\,f^2-h & 0 \\ 0 & 0 & 0 & -(h+1) \end{bmatrix} \tag{3.22}$$

and that Ricci in type $(1,1)$-form by

$$Ricci(1,1) = \frac{1}{2(h-f^2)} \begin{bmatrix} h & -f & -f^2 & 0 \\ -f & 1-h & 2f & 0 \\ 0 & f & -1 & 0 \\ 0 & 0 & 0 & -(h+1) \end{bmatrix}. \tag{3.23}$$

As regards eq.(3.22), let us consider eigenvalues recognizing that the eigenvalues are not of themselves invariant but that their signs identify the signature. As such the eigenvalues turn out to be

$$-\frac{1}{2},\ -\frac{h+1}{2(h-f^2)},\ \frac{1+2\,f^2 \pm \sqrt{1+8\,f^2+8\,f^4-4\,h-8\,hf^2+4\,h^2}}{4(h-f^2)}. \tag{3.24}$$

Clearly, the first two eigenvalues are negative. Since $Ricci(0,2)$ is symmetric the remaining two eigenvalues are also real. Hence, the third eigenvalue in the list is positive.

Now, $(1+2f^2)^2-(1+8\,f^2+8\,f^4-4\,h-8\,hf^2+4\,h^2) = 4(h-f^2)(1+f^2-h)$.

Hence the fourth eigenvalue is positive, zero or negative according as $0 < h - f^2 < 1$, $h = 1 + f^2$ or $h > 1 + f^2$ So the signature of $Ricci(0,2)$ can only be one of $(+,+,-,-)$ or $(+,-,-,0)$ or $(+,-,-,-)$.

Let us consider next the eigenvalues of Ricci as a $(1,1)$-tensor. Clearly from eq.(3.23) one eigenvalue is $-\frac{h+1}{2(h-f^2)} < 0$. The characteristic polynomial of the upper left 3×3 block of eq.(3.23) is given by

$$P(\lambda) = \lambda^3 - \frac{(3\,f^2 + 1 + h^2 - h)\,\lambda}{4\,(h - f^2)^2} - \frac{h - f^2 + 1}{8\,(-h + f^2)^2}. \tag{3.25}$$

Note that that in eq.(3.25) the coefficient of λ^2 is zero as is predicted by Corollary 2.1.2.

We remark that for the general cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d \tag{3.26}$$

the discriminant is given by [DF]

$$\triangle = 18\,abcd - 4\,b^3d + b^2c^2 - 4\,ac^3 - 27\,a^2d^2. \tag{3.27}$$

In the case of eq.(3.25) we observe that

$$a = 1, b = 0, c = -\frac{(3\,f^2 + 1 + h^2 - h)}{4\,(-h + f^2)^2}, d = -\frac{-h + f^2 + 1}{8\,(-h + f^2)^2} \tag{3.28}$$

and the discriminant is given by

$$\Delta = \frac{(h-2)^2\,(2\,h-1)^2\,(h+1)^2 + 9\,f^2x}{64(h - f^2)^6} \tag{3.29}$$

where

$$x = 9\,f^2 - 2\,h + 6\,hf^2 - 6\,h^2 + 4 + 6\,f^4 + 4\,h^4 - 3\,f^6 + 4\,h^3 - 6\,h^2f^2 + 12\,f^4h. \quad (3.30)$$

We can rewrite x as,

$$x = 3(h - f^2)^3 + 9\,f^2 + 6\,hf^2 + 6\,f^4 + 3\,h^2f^2 + 3\,f^4h + \left(h^3 - 6\,h^2 - 2\,h + 4 + 4\,h^4\right). \quad (3.31)$$

Hence

$$x > h^3 - 6\,h^2 - 2\,h + 4 + 4\,h^4. \quad (3.32)$$

Put

$$y = h^3 - 6\,h^2 - 2\,h + 4 + 4\,h^4. \quad (3.33)$$

Now we consider three cases.

Case (i) $h \geq 1$: then

$$y = 3\,(h-1)^3 + \left(h^2 - h + 1\right)^2 + 3\,(h-1)\left(h^3 + h^2 + h - 2\right) \geq 0.$$

Case (ii) $0 < h < \frac{1}{2}$: then $y = h^3 - 6\,h^2 - 2\,h + 4 + 4\,h^4$

$\geq -6\,h^2 - 2\,h + 4 = 3 - 6h^2 + 1 - 2h > 0$, since $h < \frac{1}{2}$ implies $1 - 2h > 0$ and $h^2 < \frac{1}{4}$ implies $3 - 6h^2 > 6(\frac{1}{4} - h^2) > 0$.

Case (iii) $1 > h \geq \frac{1}{2}$: then $y = h^3 - 6\,h^2 - 2\,h + 4 + 4\,h^4$

$=\left(h - \frac{1}{2}\right)^3 + \left(2\,h^2 - \frac{1}{4}\sqrt{22}\right)^2 + \frac{11}{4}(1-h) + \left(\sqrt{22} - \frac{9}{2}\right)h^2 \geq 0$, since $\sqrt{22} > 4.5$.

Therefore in all cases, $y \geq 0$ and from eq.(3.31), $x \geq 0$. Therefore from eq.(3.29), $\Delta \geq 0$. Hence $P(\lambda)$ has only real zeros and $R(1,1)$ has only real eigenvalues.

3.4 Algebra 4.2p

4.2$p(p \neq 0)$ $[e_1, e_4] = pe_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$

$$S = \begin{bmatrix} e^{pw} & 0 & 0 & x \\ 0 & e^w & we^w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^{pw}D_x, e^wD_y, e^w(D_z + wD_y), D_w$

Left-invariant one forms $e^{-pw}dx, e^{-w}(dy - wdz), e^{-w}dz, dw$

Right-invariant vector fields:$D_x, D_y, D_z, D_w + pxD_x + (y+z)D_y + zD_z$

Right-invariant one forms $dx - pxdw, dy - (y+z)dw, dz - zdw, dw.$

3.4.1 Reduction of metric

Again we will start from eq.(3.3) as the original form of the matrix of the metric. The space of derivations of algebra 4.2 is six-dimensional and is given by the following matrix:

$$der(\mathfrak{g}) = \begin{bmatrix} \phi & 0 & 0 & \gamma \\ 0 & \theta & \mu & \delta \\ 0 & 0 & \theta & \epsilon \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.34}$$

Now, we proceed to reduce the metric g using the entries in (3.34) one at a time in the following way. In (3.34) put $\mu = 1$ and the remaining entries zero to obtain a matrix M; then find $P = e^{tM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.35}$$

On the other hand

$$G_t = P^T G P = \begin{bmatrix} a & b & tb+c & d \\ b & e & te+f & g \\ tb+c & te+f & (te+f)\,t+tf+h & tg+i \\ d & g & tg+i & j \end{bmatrix}. \tag{3.36}$$

Since $e \neq 0$, we may choose $t = \frac{-f}{e}$ and hence it may be assumed $f = 0$ in (3.3).

Next, in (3.18) put $\gamma = 1$ and the remaining entries zero to obtain a matrix M; then find $P = e^{sM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.37}$$

On the other hand

$$g_s = P^T g P = \begin{bmatrix} a & b & c & sa+d \\ b & e & 0 & sb+k \\ c & 0 & h & sc+i \\ sa+d & sb+k & sc+i & as^2+2sd+j \end{bmatrix}. \tag{3.38}$$

Since $a \neq 0$, choose $s = -\frac{d}{a}$ in g_s, which is equivalent to $d = 0$ in the matrix g.

Next, in (3.18) put $\delta = 1$ and the remaining entries zero to obtain a matrix M; then find $P = e^{rM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.39}$$

On the other hand

$$g_r = P^T g P = \begin{bmatrix} a & b & c & rb \\ b & e & 0 & re+k \\ c & 0 & h & i \\ rb & re+k & i & r^2e+2rk+j \end{bmatrix}. \tag{3.40}$$

Since $e \neq 0$, choose $r = -\frac{k}{e}$ in g_r which is equivalent to $k = 0$ in g.

Next, in (3.18) put $\epsilon = 1$ and the remaining entries zero to obtain a matrix M; then find $P = e^{pM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.41}$$

On the other hand

$$g_p = P^T g P = \begin{bmatrix} a & b & c & pc \\ b & e & 0 & 0 \\ c & 0 & h & ph+i \\ pc & 0 & ph+i & (ph+i)\,p+pi+j \end{bmatrix}. \tag{3.42}$$

Since $h \neq 0$, choose $p = -\frac{i}{h}$ in g_i which is equivalent to $i = 0$ in the matrix g. Next, in (3.18) put $\phi = 1$ and the remaining entries zero to obtain a matrix M; then find $P = e^{tM}$. Then P acts as an automorphism of $\mathfrak{g}$ and is given by

$$P = \begin{bmatrix} \mathrm{e}^t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.43}$$

Again put $\theta = 1$ and the remaining entries zero in eq.(3.1) so as to obtain a matrix

M' so that we get another automorphism as,

$$P' = e^{wM'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{w} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{w} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.44}$$

Now P and P' act on the metric g to give $g_{t,w}$ by

$$g_{t,w} = PP'^{T} g(PP') = \begin{bmatrix} \mathrm{e}^{2\,t}a & \mathrm{e}^{t+w}b & \mathrm{e}^{t+w}c & 0 \\ \mathrm{e}^{t+w}b & \mathrm{e}^{2\,w}e & 0 & 0 \\ \mathrm{e}^{t+w}c & 0 & \mathrm{e}^{2\,w}h & 0 \\ 0 & 0 & 0 & j \end{bmatrix}. \tag{3.45}$$

Finally, by scaling, we can multiply $g_{t,w}$ by e^{v} then we get,

$$g_{t,w,v} = \begin{bmatrix} \mathrm{e}^{v+2\,t}a & \mathrm{e}^{t+v+w}b & \mathrm{e}^{t+v+w}c & 0 \\ \mathrm{e}^{t+v+w}b & \mathrm{e}^{v+2\,w}e & 0 & 0 \\ \mathrm{e}^{t+v+w}c & 0 & \mathrm{e}^{v+2\,w}h & 0 \\ 0 & 0 & 0 & \mathrm{e}^{v}j \end{bmatrix}. \tag{3.46}$$

Since $a, e, j > 0$ we may choose $\mathrm{e}^{v+2t} = a^{-1}$, $\mathrm{e}^{v+2w} = e^{-1}$ and $\mathrm{e}^{v} = j^{-1}$, which is equivalent to $a = 1$, $e = 1$ and $j = 1$. So

$$g = \begin{bmatrix} 1 & b & c & 0 \\ b & 1 & 0 & 0 \\ c & 0 & h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.47}$$

where $det(g) = h - hb^2 - c^2 > 0$ and $h > 0$.

3.4.2 Ricci tensor

$$Ricci(0,2) = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & R_{44} \end{bmatrix} \tag{3.48}$$

where

$R_{11} = \frac{2\,bcp+4\,hp-2\,b^2hp-2\,c^2p-c^2-b^2-2\,bc-b^2h+2\,hp^2-3\,c^2p^2-3\,b^2hp^2+b^4}{-2(h-b^2h-c^2)}$

$R_{12} = \frac{b+c-5\,bhp-cp+b^2c+2\,bc^2-bh-b^3-b^2cp+2\,b^3h+b^3hp^2+3\,b^3hp+bc^2p^2+3\,bc^2p}{2(h-b^2h-c^2)}$

$R_{13} = \frac{cp^2b^2h+3\,cpb^2h-ch-2\,bhp+2\,bc^2-bh+2\,c^3+c^3p^2+3\,c^3p+b^3h-5\,chp+2\,cb^2h+2\,b^3hp+bc^2p}{2(h-b^2h-c^2)}$

$R_{22} = \frac{-4\,h+1-2\,bcp-2\,hp+2\,c^2p+4\,c^2-b^2+2\,bc+5\,b^2h+b^2hp^2}{2(h-b^2h-c^2)}$

$R_{23} = \frac{-hp+b^2hp+2\,b^2h+3\,c^2+cp^2bh-2\,bhcp+bch-2\,h}{2(h-b^2h-c^2)}$

$R_{33} = \frac{-2\,h^2p+2\,h^2pb^2+4\,b^2h^2+5\,c^2h+hp^2c^2-4\,h^2+b^2h-h+c^2}{2(h-b^2h-c^2)}$

$R_{44} = \frac{-4\,h+2\,bcp+2\,b^2hp+2\,c^2p+3\,c^2+b^2-2\,bc+3\,b^2h-1-2\,hp^2+c^2p^2+b^2hp^2}{2(h-b^2h-c^2)}$

$R_{21} = R_{12},\ R_{31} = R_{13},\ R_{32} = R_{23}.$

For the Ricci(1,1) tensor:

$$Ricci(1,1) = \begin{bmatrix} R^1_1 & R^1_2 & R^1_3 & 0 \\ R^2_1 & R^2_2 & R^2_3 & 0 \\ R^3_1 & R^3_2 & R^3_3 & 0 \\ 0 & 0 & 0 & R^4_4 \end{bmatrix} \tag{3.49}$$

where

$R^1_1 = \frac{b^2hp^2+3\,b^2hp+2\,b^2h+bcp+2\,bc-2\,hp^2-4\,hp+3\,c^2p+c^2p^2+2\,c^2}{2(h-b^2h-c^2)}$

$R^2_1 = \frac{b^3+b^2c+2\,b^2cp+bhp+bh-2\,bhp^2-b+cp-c}{2(-h+b^2h+c^2)}$

$R^3_1 = \frac{2\,b^3p+b^3-2\,bp-b+2\,cp^2-c-cp}{2(h-b^2h-c^2)}$

$R^1_2 = -\frac{3(bhp-bh-c)}{2(h-b^2h-c^2)}$

$R^2_2 = \frac{-b^2+b^2hp^2+3\,b^2hp+2\,b^2h-bc-2\,bcp-4\,h+1-2\,hp+2\,c^2p+4\,c^2}{2(h-b^2h-c^2)}$

$R^3_2 = \frac{b^2p+2\,b^2+bcp-2\,bc+cp^2b-p-2}{2(h-b^2h-c^2)}$

$R^1_3 = \frac{bhp-bh-3\,ch+3\,chp-c}{2(-h+b^2h+c^2)}$

$R^2_3 = \frac{2\,b^2hp+b^2h+bhcp-2\,bhc+cp^2bh-bc-hp-2\,h+3\,c^2}{2(h-b^2h-c^2)}$

$R^3_3 = \frac{4\,b^2h+2\,b^2hp+b^2+bcp-bc-2\,hp-4\,h+2\,c^2-1+3\,c^2p+c^2p^2}{2(h-b^2h-c^2)}$

$R^4_4 = \frac{-4\,h+2\,bcp+2\,b^2hp+2\,c^2p+3\,c^2+b^2-2\,bc+3\,b^2h-1-2\,hp^2+c^2p^2+b^2hp^2}{2(h-b^2h-c^2)}.$

3.5 **Algebra** 4.3

4.3 $[e_1, e_4] = e_1, [e_3, e_4] = e_2$

$$S = \begin{bmatrix} e^w & 0 & 0 & x \\ 0 & 1 & w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^w D_x, D_y, D_z + wD_y, D_w$

Left-invariant one forms $e^{-w}dx, dy - wdz, dz, dw$

Right-invariant vector fields $D_x, D_y, D_z, D_w + xD_x + zD_y$

Right-invariant one forms $dx - xdw, dy - zdw, dz, dw$.

3.5.1 Reduction of metric

It turns out that the space of derivations is precisely the same as the generic case for Algebra 4.2 and hence the metric can be reduced to the same form as in that case.

Ricci(0,2) is given by

$$\frac{1}{2(h - b^2h - c^2)} \begin{bmatrix} 3\,b^2h-2\,cb+3\,c^2+b^2-b^4-2\,h & b^3h+bc^2-cb^2-c+b-b^3 & cb^2h+c^3+bc^2-2\,bh+2\,b^3h & 0 \\ b^3h+bc^2-cb^2-c+b-b^3 & b^2h-2\,cb-b^2+1 & \left(cb+b^2-1\right)h & 0 \\ cb^2h+c^3+bc^2-2\,bh+2\,b^3h & \left(cb+b^2-1\right)h & hc^2+b^2h-h+c^2 & 0 \\ 0 & 0 & 0 & R_{44} \end{bmatrix} \tag{3.50}$$

where $R_{44} = b^2h + 2\,cb - 2\,h + c^2 + b^2 - 1$ and Ricci(1,1) by

$$\frac{-1}{2(h - hb^2 - c^2)} \begin{bmatrix} -2\,h+b^2h+c^2+cb & 0 & c-bh & 0 \\ -b^3-2\,cb^2+2\,bh+b-c & b^2h-2\,cb-b^2+1 & 2\,b^2h+bhc-cb-h & 0 \\ 2\,c-2\,b+2\,b^3 & cb+b^2-1 & b^2+cb+c^2-1 & 0 \\ 0 & 0 & 0 & b^2h+2\,cb-2\,h+c^2+b^2-1 \end{bmatrix}. \tag{3.51}$$

3.6 **Algebra** 4.4

4.4 $[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$

$$S = \begin{bmatrix} e^w & we^w & \frac{w^2}{2}e^w & x \\ 0 & e^w & we^w & y \\ 0 & 0 & e^w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^w D_x, e^w(D_y + wD_x), e^w(D_z + wD_y + \frac{w^2}{2}D_x), D_w$

Left-invariant one forms $e^{-w}(dx - wdy + \frac{w^2}{2}dz), e^{-w}(dy - wdz), e^{-w}dz, dw$

Right-invariant vector fields $D_x, D_y, D_z, D_w + (x+y)D_x + (y+z)D_y + zD_z$

Right-invariant one forms $dx - (x+y)dw, dy + (y+z)dw, dz - zdw, dw$.

3.6.1 Reduction of metric

The space of derivations is given by $der(\mathfrak{g}) = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & \beta & \epsilon \\ 0 & 0 & \alpha & \phi \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & f & 0 \\ 0 & f & h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.52}$$

where $e > 0, eh - f^2 > 0$.

3.6.2 Ricci tensor

The Ricci (0,2)-tensor is given by

$$\frac{1}{2(h-f^2)}\begin{bmatrix} h-6\,eh+6\,f^2 & -fe+3\,f^2-3\,eh & -f^2 & 0 \\ -fe+3\,f^2-3\,eh & -eh+f^2-6\,e^2h+6\,ef^2+e^3 & -3\,e^2h+3\,ef^2-6\,feh+6\,f^3+fe^2 & 0 \\ -f^2 & -3\,e^2h+3\,ef^2-6\,feh+6\,f^3+fe^2 & -6\,feh+6\,f^3-6\,eh^2+6\,hf^2-e^2h+2\,ef^2 & 0 \\ 0 & 0 & 0 & -6\,eh+6\,f^2-h-e^2 \end{bmatrix} \tag{3.53}$$

and Ricci(1,1) by

$$R = \frac{1}{2(eh-f^2)}\begin{bmatrix} h-6\,eh+6\,f^2 & -fe+3\,f^2-3\,eh & -f^2 & 0 \\ -f-3\,h & e^2-6\,eh+3\,fe-h+6\,f^2 & -3\,eh+2\,fe+6\,f^2 & 0 \\ 3\,f & -3\,e^2+f & -e^2-6\,eh-3\,fe+6\,f^2 & 0 \\ 0 & 0 & 0 & -6\,eh+6\,f^2-h-e^2 \end{bmatrix}. \tag{3.54}$$

3.7 Algebra $4.5ab$

$4.5ab$ $(ab \neq 0, -1 \leq a \leq b \leq 1)$ $[e_1, e_4] = ae_1, [e_2, e_4] = be_2, [e_3, e_4] = ce_3$

$$S = \begin{bmatrix} e^{aw} & 0 & 0 & x \\ 0 & e^{bw} & 0 & y \\ 0 & 0 & e^{cw} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields $e^{aw}D_x, e^{bw}D_y, e^{cw}D_z, D_w$

Left-invariant one forms $e^{-aw}dx, e^{-bw}dy, e^{-cw}dz, dw$

Right-invariant vector fields $D_x, D_y, D_z, D_w + axD_x + byD_y + czD_z$

Right-invariant one forms $dx - axdw, dy - bydw, dz - czdw, dw.$

3.7.1 Reduction of metric

The space of derivations is given by $\begin{bmatrix} \alpha & 0 & 0 & \delta \\ 0 & \beta & 0 & \epsilon \\ 0 & 0 & \gamma & \phi \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} 1 & f & e & 0 \\ f & 1 & d & 0 \\ e & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.55}$$

where $1 - d^2 - e^2 - f^2 + 2def > 0, 1 - d^2 > 0, 1 - e^2 > 0, 1 - f^2 > 0$.

3.7.2 Ricci tensor

We find that

$$Ricci(0,2) = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & R_{44} \end{bmatrix} \tag{3.56}$$

where

$$
\begin{aligned}
R_{11} &= -\frac{1}{\Delta}(2+2\,p+2\,q+2\,fbcp+2\,fbcq-2\,f^2q-2\,pc^2-2\,pf^2+6\,fbc-3\,b^2-3\,c^2-2\,f^2 \\
&\quad +2\,fbcpq-2\,qb^2-2\,pb^2c^2q+b^2p^2c^2+c^2b^2q^2-c^2q^2-b^2p^2) \\
R_{12} &= \frac{1}{\Delta}(3\,pbf^2+2\,bf^2q-bq+b^3+fcp-fcq-2\,fb^2c-4\,pb+c^2b-cpb^2fq+3\,pc^2b-fb^2q^2c-fcpq \\
&\quad -5\,cpb^2f-fb^2cq-2\,fb^2cp^2+2\,pc^2bq+bf^2p^2-bqp+fcq^2+2\,b^3p+b^3p^2+b^3q+b^3qp) \\
R_{13} &= \frac{1}{\Delta}(-fbp-cp-4\,cq+fbq+cb^2-pc^2bfq+3\,cf^2q+2\,cf^2p+c^3p+2\,c^3q+3\,cb^2q-fbpq-pc^2bf \\
&\quad -5\,fbc^2qa+2\,cb^2qp-2\,fbc^2q^2+c^3+fbp^2+cf^2q^2-2\,fbc^2-cqp+c^3qp+c^3q^2-fp^2c^2b) \\
R_{22} &= -\frac{1}{\Delta}(2\,p+2\,fbcp+2\,fbcq-2\,pc^2-2\,pf^2-b^2+2\,fbcpq+6\,fbcp^2-f^2q^2+2\,qp-2\,b^2f^2q \\
&\quad -2\,qb^2p-2\,pc^2q+2\,p^2+b^2f^2-3\,b^2p^2-2\,p^2c^2-3\,p^2f^2+b^2f^2q^2) \\
R_{23} &= \frac{1}{\Delta}(bc-cpb-cbq-fp-fq-5\,cpbf^2q+2\,fb^2q+f^3q-cpbf^2-bf^2cq+3\,pc^2fq+cbqp-2\,bf^2cq^2 \\
&\quad +3\,pb^2fq-2\,bf^2cp^2+2\,pc^2f+f^3q^2-bf^2c+fb^2p^2-4\,fpq+2\,pf^3q+c^2fq^2+f^3p^2+pf^3) \\
R_{33} &= \frac{1}{\Delta}(-2\,q-2\,fbcp-2\,fbcq+2\,f^2q-f^2c^2+c^2-2\,fbcpq+2\,qb^2-6\,fbcq^2-2\,q^2+3\,f^2q^2 \\
&\quad -2\,qp+2\,qb^2p-f^2c^2p^2+2\,f^2c^2p+2\,pc^2q+3\,c^2q^2+p^2f^2+2\,b^2q^2) \\
R_{44} &= \frac{1}{\Delta}(-2-2\,fbcp-2\,fbcq+2\,pb^2+2\,c^2q-2\,fbc+b^2+c^2+2\,f^2-2\,fbcpq-2\,fbcq^2-2\,fbcp^2 \\
&\quad -2\,q^2+f^2q^2+2\,pf^2q-2\,p^2+c^2q^2+b^2p^2+2\,p^2c^2+p^2f^2+2\,b^2q^2)
\end{aligned}
$$

and

$$R_{21}=R_{12},\; R_{31}=R_{13},\; R_{32}=R_{23},\; \Delta=2j(1-b^2-c^2-f^2+2\,fbc).$$

The Ricci(1,1) form is given by:

$$
R(1,1)=\begin{bmatrix} R^1_1 & R^1_2 & R^1_3 & 0 \\ R^2_1 & R^2_2 & R^2_3 & 0 \\ R^3_1 & R^3_2 & R^3_3 & 0 \\ 0 & 0 & 0 & R^4_4 \end{bmatrix} \tag{3.57}
$$

where

$$
\begin{aligned}
R_1^1 &= \frac{1}{\Delta}(-2-2\,q-2\,p+2\,f^2+c^2+b^2+2\,f^2q+2\,pf^2+2\,c^2q+pc^2-2\,fbc+b^2q+2\,b^2p \\
&\quad +c^2q^2+pc^2q-3\,fbcq-3\,fbcp+b^2qp+b^2p^2-fbcq^2-2\,fbcpq-fbcp^2) \\
R_2^1 &= \frac{1}{\Delta}(-c^2bq^2+2\,bpc^2-2\,c^2bq+2\,b+bpc^2q-2\,bp+bq-bqp+cfq^2-2\,cf+cfq) \\
R_3^1 &= -\frac{1}{\Delta}(-2\,cb^2q+cb^2p^2-cb^2qp+2\,cb^2p-fbp^2+2\,fb-fbp-2\,c-pc+2\,cq+cqp) \\
R_1^2 &= -\frac{1}{\Delta}(-bf^2q+2\,bpf^2q+bf^2q^2-2\,bpf^2+2\,bp-2\,bp^2+bq-bqp-cfq^2-pcfq+2\,cfp^2) \\
R_2^2 &= \frac{1}{\Delta}(-2\,p-2\,qp-2\,p^2+b^2+f^2q+pf^2+2\,pc^2-fbc+b^2q+2\,b^2p+f^2q^2+2\,pf^2q \\
&\quad +p^2f^2+2\,pc^2q+2\,p^2c^2-2\,fbcq-3\,fbcp+b^2qp+b^2p^2-fbcq^2-3\,fbcpq-2\,fbcp^2) \\
R_3^2 &= \frac{1}{\Delta}(2\,b^2pfq-fb^2+fb^2q-2\,b^2pf-2\,cbp^2+pcb+cb+fp-fq-2\,fpq+2\,fp^2) \\
R_1^3 &= \frac{1}{\Delta}(-2\,fbq^2+fbp^2+fbpq+2\,cf^2q+cqp+cf^2p-pc-2\,cq-2\,cqpf^2+2\,cq^2-cf^2p^2) \\
R_2^3 &= -\frac{1}{\Delta}(2\,cbq^2-cb-cbq+2\,fpq-fq+fc^2-fpc^2+2\,fc^2q-2\,fq^2-2\,fpc^2q+fp) \\
R_3^3 &= \frac{1}{\Delta}(-2\,q-2\,q^2-2\,qp+c^2+f^2q+pf^2+2\,c^2q+pc^2-fbc+2\,b^2q+f^2q^2+2\,pf^2q \\
&\quad +p^2f^2+c^2q^2+pc^2q-3\,fbcq-2\,fbcp+2\,b^2q^2+2\,b^2qp-2\,fbcq^2-3\,fbcpq-fbcp^2) \\
R_4^4 &= \frac{1}{\Delta}(-2-2\,fbcp-2\,fbcq+2\,b^2p+2\,c^2q-2\,fbc+b^2+c^2+2\,f^2-2\,fbcpq-2\,fbcq^2 \\
&\quad -2\,fbcp^2-2\,q^2+f^2q^2+2\,pf^2q-2\,p^2+c^2q^2+b^2p^2+2\,p^2c^2+p^2f^2+2\,b^2q^2).
\end{aligned}
$$

3.8 Algebra $4.6ab$

$4.6ab$ $(a \neq 0, b \geq 0)$ $[e_1, e_4] = ae_1, [e_2, e_4] = be_2 - e_3, [e_3, e_4] = e_2 + be_3$

$$
S = \begin{bmatrix} e^{aw} & 0 & 0 & x \\ 0 & e^{bw}\cos w & e^{bw}\sin w & y \\ 0 & -e^{bw}\sin w & e^{bw}\cos w & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Left-invariant vector fields $e^{aw}D_x, e^{bw}(\cos wD_y - \sin wD_z), e^{bw}(\sin wD_y + \cos wD_z), D_w$

Left-invariant one forms $e^{-aw}dx, e^{-bw}(\cos wdy - \sin wdz), e^{-bw}(\sin wdy + \cos wdz), dw$

Right-invariant vector fields $D_x, D_y, D_z, D_w + axD_x + (by + z)D_y + (bz - y)D_z$

Right-invariant one forms $dx - axdw, dy - (by + z)dz, dz - (bz - y)dy, dw.$

3.8.1 Reduction of metric

The space of derivations is given by $der(\mathfrak{g}) = \begin{bmatrix} \alpha & 0 & 0 & \delta \\ 0 & \beta & \gamma & \epsilon \\ 0 & -\gamma & \beta & \phi \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$g = \begin{bmatrix} 1 & c & d & 0 \\ c & 1 & 0 & 0 \\ d & 0 & h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.58}$$

with $c^2 < 1, h - hc^2 - d^2 > 0$.

3.8.2 Ricci tensor

We find that

$$Ricci(0,2) = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & R_{44} \end{bmatrix} \tag{3.59}$$

where

$$
\begin{aligned}
R_{11} &= \frac{1}{\Delta}(2\,chda + 2\,cdb - 2\,c^2d^2 + c^2 - 2\,chdb + d^2h - 2\,cda - d^4 + c^2hb^2 + 3\,c^2a^2h \\
&\quad +2\,c^2bha - 2\,a^2h + 3\,a^2d^2 + b^2d^2 + 2\,ad^2b - 4\,bha - c^4) \\
R_{12} &= \frac{1}{\Delta}(c + dhb + 3\,c^3ahb - c^2ahd + 3\,cad^2b - 5\,chba - 2\,c^2bhd - da + db - cd^2 - c^2da \\
&\quad +c^2db - c^3 - chb^2 - 2\,d^3a - d^3b + c^3a^2h + ca^2d^2 + 2\,c^3b^2h + 2\,cb^2d^2 + 2\,dha) \\
R_{13} &= \frac{1}{\Delta}(-chb - 2\,cha + 3\,c^2ahdb + cad^2 + 2\,cd^2b - hd^3 - ch^2b + da^2c^2h + cad^2h - 5\,dhba \\
&\quad +2\,db^2c^2h - cd^2bh + h^2d + d^3a^2 + 2\,d^3b^2 + h^2ac - db^2h + 3\,d^3ab + 2\,c^3ah + c^3hb - c^2hd) \\
R_{22} &= \frac{1}{\Delta}(1 + 2\,cdb - c^2 - h^2 + d^2h - 2\,cda + 5\,c^2hb^2 + c^2a^2h - 4\,b^2h + 4\,b^2d^2 + c^2h^2 + 2\,ad^2b - 2\,bha) \\
R_{23} &= \frac{1}{\Delta}(-ha + c^2ah + 2\,c^2bh - chd + 3\,d^2b + h^2a - ad^2h + da^2hc - 3\,c^2h^2b - 2\,chdab \\
&\quad +db^2ch - 2\,bh + 2\,h^2b - 2\,d^2bh) \\
R_{33} &= \frac{1}{\Delta}(-h + c^2h + d^2 + h^3 - h^2d^2 - 4\,h^2b^2 + 4\,c^2h^2b^2 + 5\,b^2hd^2 - 2\,h^2ab + a^2hd^2 \\
&\quad +2\,h^2ac^2b + 2\,h^2acd - 2\,h^2dcb) \\
R_{44} &= \frac{1}{\Delta}(2\,h - 2\,chda - 2\,cdb - 2\,d^2 + c^2 - 2\,c^2h + 2\,chdb - 1 - h^2 + d^2h + 2\,cda + 3\,c^2hb^2 + c^2a^2h \\
&\quad +2\,c^2bha - 2\,a^2h + a^2d^2 - 4\,b^2h + 3\,b^2d^2 + 2\,ad^2b)
\end{aligned}
$$

and

$$R_{21} = R_{12},\; R_{31} = R_{13},\; R_{32} = R_{23},\; \Delta = 2(h - c^2h - d^2).$$

The Ricci(1,1) form is given by:

$$
R(1,1) = \begin{bmatrix} R^1_1 & R^1_2 & R^1_3 & 0 \\ R^2_1 & R^2_2 & R^2_3 & 0 \\ R^3_1 & R^3_2 & R^3_3 & 0 \\ 0 & 0 & 0 & R^4_4 \end{bmatrix} \tag{3.60}
$$

where

$$
\begin{aligned}
R^1_1 &= \frac{1}{\Delta}(2\,c^2hb^2+3\,c^2bha+c^2a^2h-2\,chdb-chda+2\,cdb+cda-2\,a^2h-4\,bha+3\,ad^2b+2\,b^2d^2+a^2d^2)\\
R^1_2 &= \frac{1}{\Delta}(ch^2-3\,chba+3\,chb^2-dhb+dha+3\,db)\\
R^1_3 &= -\frac{1}{\Delta}(-3\,ch^2b-cha+chb+3\,db^2h-3\,dhba+d)\\
R^2_1 &= -\frac{1}{\Delta}(-c^3-c^2db-2\,c^2da-chba-chb^2+2\,ca^2h+c-cd^2+db+dhb-d^3b-da+2\,dha-2\,d^3a)\\
R^2_2 &= \frac{1}{\Delta}(-c^2+2\,c^2hb^2+3\,c^2bha+c^2a^2h+chdb-chda-cdb-2\,cda+1+4\,b^2d^2-h^2+d^2h\\
&\quad +2\,ad^2b-4\,b^2h-2\,bha)\\
R^2_3 &= \frac{1}{\Delta}(2\,c^2ah+c^2bh-2\,db^2ch-chd+chdab+da^2hc-cd-ha+h^2a-ad^2h-2\,bh+3\,d^2b-2\,d^2bh+2\,h^2b)\\
R^3_1 &= \frac{1}{\Delta}(2\,c^3a+bc^3-c^2d+cha-chb-2\,ca-cb+cd^2b+2\,cad^2+dh-dba+2\,da^2-db^2-d^3)\\
R^3_2 &= -\frac{1}{\Delta}(-3\,c^2bh+2\,c^2b+c^2a-chd-2\,db^2c-cd+cdab+ca^2d+2\,bh+ha-2\,b-2\,ad^2-a-d^2b)\\
R^3_3 &= \frac{1}{\Delta}(4\,c^2hb^2+2\,c^2bha+c^2+chdb+2\,chda-cdb+cda+h^2-4\,b^2h-d^2h-2\,bha+3\,ad^2b\\
&\quad -1+a^2d^2+2\,b^2d^2)\\
R^4_4 &= \frac{1}{\Delta}(2\,h-2\,chda-2\,cdb-2\,d^2+c^2-2\,c^2h+2\,chdb-1-h^2+d^2h+2\,cda+3\,c^2hb^2\\
&\quad +c^2a^2h+2\,c^2bha-2\,a^2h+a^2d^2-4\,b^2h+3\,b^2d^2+2\,ad^2b).
\end{aligned}
$$

3.9 Algebra 4.7

4.7 $[e_2,e_3]=e_1,[e_1,e_4]=2e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_2+e_3$

$$
S=\begin{bmatrix} e^{2w} & -ze^w & (y-zw)e^w & x\\ 0 & e^w & we^w & y\\ 0 & 0 & e^w & z\\ 0 & 0 & 0 & 1\end{bmatrix}.
$$

Left-invariant vector fields $\frac{1}{2}e^{2w}D_x, e^w(D_y-zD_x), e^w(D_z+wD_y+(y-zw)D_x), -D_w$

Left-invariant one forms $e^{-2w}(dx+zdy-ydz), e^{-w}(dy-wdz), e^{-w}dz, dw$

Right-invariant vector fields $-\frac{1}{2}D_x, zD_x+D_y, D_z-yD_x, D_w+2xD_x+(y+z)D_y+zD_z$

Right-invariant one forms $dx-zdy+ydz+(z^2-2x)dw, dy-(y+z)dw, dz-zdw, dw$.

3.9.1 Reduction of metric

The space of derivations is given by $der(\mathfrak{g}) = \begin{bmatrix} 2\alpha & \beta & \gamma & \delta \\ 0 & \alpha & \epsilon & \gamma-\beta \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} a & b & c & 0 \\ b & e & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.61}$$

where $a > 0, e > 0, ae - b^2 - c^2e > 0$.

3.9.2 Ricci tensor

$$Ricci(0,2) = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ 0 & R_{42} & R_{43} & R_{44} \end{bmatrix} \tag{3.62}$$

where

$$R_{21} = R_{12},\ R_{31} = R_{13},\ R_{32} = R_{23},\ R_{42} = R_{24},\ R_{43} = R_{34}$$

and

$$
\begin{aligned}
R_{11} &= \frac{a^3 - 16\,a^2e + 17\,c^2ae - 2\,bcae + 17\,b^2a + aeb^2 - b^4}{2(ae - b^2 - c^2e)}\\
R_{12} &= \frac{a^2b - ae^2c - 11\,aeb + 12\,ec^2b - b^2ec + 12\,b^3 + ae^2b - b^3e}{2(ae - b^2 - c^2e)}\\
R_{13} &= \frac{ca^2 - 5\,aeb - 11\,cae + 4\,ec^2b + 12\,c^3e + 12\,cb^2 + 5\,b^3}{2(ae - b^2 - c^2e)}\\
R_{22} &= \frac{-a^2e - 8\,ae^2 + 8\,c^2e^2 - 2\,bce^2 + 9\,eb^2 + ae^3 + 2\,b^2a - b^2e^2}{2(ae - b^2 - c^2e)}\\
R_{23} &= \frac{-4\,ae^2 + 3\,c^2e^2 + 4\,eb^2 + ecb + 2\,abc}{2(ae - b^2 - c^2e)}\\
R_{24} &= \frac{-cae + aeb - b^3}{2(ae - b^2 - c^2e)}\\
R_{33} &= \frac{-a^2 - 8\,ae + 9\,c^2e + 8\,b^2 - ae^2 + 2\,c^2a + c^2e^2 + eb^2}{2(ae - b^2 - c^2e)}\\
R_{34} &= \frac{b\,(a - cb)}{2(ae - b^2 - c^2e)}\\
R_{44} &= \frac{-12\,ae + 11\,c^2e + 2\,ecb + 11\,b^2 - ae^2 + eb^2}{2(ae - b^2 - c^2e)}.
\end{aligned}
$$

The Ricci(1,1) form is given by:

$$
R(1,1) = \begin{bmatrix} R^1_1 & R^1_2 & R^1_3 & R^1_4 \\ R^2_1 & R^2_2 & R^2_3 & R^2_4 \\ R^3_1 & R^3_2 & R^3_3 & R^3_4 \\ 0 & R^4_2 & R^4_3 & R^4_4 \end{bmatrix} \tag{3.63}
$$

where

$$
\begin{aligned}
R_1^1 &= \frac{a^2 - 16\,ae + 12\,c^2e + 4\,ecb + 12\,b^2}{2(ae - b^2 - c^2e)} \\
R_2^1 &= \frac{2\,ab + 3\,ce^2 - 3\,eb}{2(ae - b^2 - c^2e)} \\
R_3^1 &= \frac{2\,ca + ce^2 - 3\,ec - eb}{2(ae - b^2 - c^2e)} \\
R_4^1 &= \frac{-b^2}{2(ae - b^2 - c^2e)} \\
R_1^2 &= \frac{-cae + aeb + 5\,ab - b^3 - 5\,cb^2}{2(ae - b^2 - c^2e)} \\
R_2^2 &= \frac{-a^2 + ae^2 - 8\,ae + 8\,c^2e - 5\,ecb - eb^2 + 12\,b^2}{2(ae - b^2 - c^2e)} \\
R_3^2 &= \frac{-4\,ae - ecb + 3\,c^2e + 4\,cb + 5\,b^2}{2(ae - b^2 - c^2e)} \\
R_4^2 &= \frac{a\,(-c + b)}{2(ae - b^2 - c^2e)} \\
R_1^3 &= \frac{5\,cae - 5\,aeb + 5\,b^3}{2(ae - b^2 - c^2e)} \\
R_2^3 &= \frac{-4\,e\left(-cb + ae - b^2\right)}{2(ae - b^2 - c^2e)} \\
R_3^3 &= \frac{-a^2 - ae^2 - 8\,ae + 12\,c^2e + ecb + eb^2 + 8\,b^2}{2(ae - b^2 - c^2e)} \\
R_4^3 &= \frac{ab}{2(ae - b^2 - c^2e)} \\
R_2^4 &= \frac{-cae + aeb - b^3}{2(ae - b^2 - c^2e)} \\
R_3^4 &= \frac{b\,(a - cb)}{2(ae - b^2 - c^2e)} \\
R_4^4 &= \frac{-12\,ae + 11\,c^2e + 2\,ecb + 11\,b^2 - ae^2 + eb^2}{2(ae - b^2 - c^2e)}.
\end{aligned}
$$

3.10 Algebra $4.8(b = -1), 4.9b(-1 < b \leq 1)$

$4.8/4.9b[e_2, e_3] = e_1, [e_1, e_4] = (b + 1)e_1, [e_2, e_4] = e_2, [e_3, e_4] = be_3$

$$
S = \begin{bmatrix} \mathrm{e}^{(b+1)w} & y\mathrm{e}^{bw} & x \\ 0 & \mathrm{e}^{bw} & z \\ 0 & 0 & 1 \end{bmatrix}.
$$

Left-invariant vector fields $e^{(b+1)w}D_x, e^w D_y, e^{bw}(D_z + yD_x), -D_w$

Left-invariant one forms $e^{-(b+1)w}(dx - ydz), e^{-w}dy, e^{-bw}dz, -dw$

Right-invariant vector fields $D_x, zD_x + D_y, -D_z, D_w + (xb + x)D_x + yD_y + bzD_z$

Right-invariant one forms $dx - zdy + (yz - (b+1)x)dw, dy - ydw, -(dz - bzdw), dw.$

3.10.1 Reduction of metric

The space of derivations is given by $der(\mathfrak{g}) = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \epsilon & 0 & \gamma \\ 0 & 0 & \alpha-\epsilon & -b\beta \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & f & k \\ 0 & f & 1 & m \\ 0 & k & m & 1 \end{bmatrix} \tag{3.64}$$

where $a > 0, 1 - f^2 > 0, 1 - m^2 + 2km - f^2 - k^2 > 0.$

3.10.2 Ricci Tensor

The R(0,2) tensor has following components:

$$\begin{aligned}
R_{11} &= \frac{1}{\Delta}(a\,(4\,f^2b^2 + 8\,f^2b + 4\,f^2 - 4 - 8\,b + a - 4\,b^2)) \\
R_{12} &= \frac{1}{\Delta}(-a\,(2\,fkb + 3\,fk - 3\,mb - 2\,m)) \\
R_{13} &= \frac{1}{\Delta}(a\,(3\,fmb + 2\,fm - 3\,k - 2\,kb) \\
R_{14} &= \frac{1}{\Delta}(akm\,(-1 + b)) \\
R_{22} &= \frac{1}{\Delta}(-a + ak^2 - 4\,b - 4 + 2\,f^2b + 5\,f^2 + f^2b^2) \\
R_{23} &= \frac{1}{\Delta}(-af + akm - fb^2 - 6\,fb - f + 2\,f^3b^2 + 4\,f^3b + 2\,f^3) \\
R_{24} &= \frac{1}{\Delta}(3\,fm - 4\,kb - 4\,k - mb^2f + 2\,f^2kb^2 + 4\,f^2kb + 2\,f^2k - 2\,fmb) \\
R_{33} &= \frac{1}{\Delta}(-a + am^2 - 4\,b^2 - 4\,b + 5\,f^2b^2 + 2\,f^2b + f^2) \\
R_{34} &= \frac{1}{\Delta}(2\,f^2mb^2 + 4\,f^2mb + 2\,f^2m + 3\,fkb^2 - fk - 4\,mb^2 - 4\,mb - 2\,fkb) \\
R_{44} &= \frac{1}{\Delta}(-4\,b^2 + 3\,f^2b^2 + 4\,k^2b^2 - 2\,mb^2fk - 4\,b + 6\,f^2b - 4\,fkmb - 4 + 3\,f^2 + 4\,m^2 - 2\,fkm)
\end{aligned}$$

and

$$R_{21}=R_{12},\, R_{31}=R_{13},\, R_{41}=R_{14},\, R_{32}=R_{23},\, R_{42}=R_{24},\, R_{43}=R_{34},\, \Delta=2(1-f^2-k^2-m^2+2fkm).$$

The Ricci(1,1) form is given by following:

$$\begin{aligned}
R_1^1 &= \frac{1}{\Delta}(4\,f^2b^2+8\,f^2b+4\,f^2-4-8\,b+a-4\,b^2)\\
R_2^1 &= \frac{1}{\Delta}(-2\,fkb-3\,fk+3\,mb+2\,m)\\
R_3^1 &= \frac{1}{\Delta}(3\,fmb+2\,fm-3\,k-2\,kb)\\
R_4^1 &= \frac{1}{\Delta}(km\,(-1+b))\\
R_1^2 &= \frac{1}{\Delta}(am\,(3\,b+2))\\
R_2^2 &= \frac{1}{\Delta}(2\,f^2b^2+4\,f^2b+2\,f^2-4-4\,b-a)\\
R_3^2 &= \frac{1}{\Delta}(f\left(-1+3\,b^2-2\,b\right))\\
R_4^2 &= \frac{1}{\Delta}(-mb^2f-2\,fmb-fm+4\,kb^2)\\
R_1^3 &= \frac{1}{\Delta}(-ak\,(2\,b+3))\\
R_2^3 &= \frac{1}{\Delta}(-f\left(-3+b^2+2\,b\right))\\
R_3^3 &= \frac{1}{\Delta}(2\,f^2b^2+4\,f^2b+2\,f^2-4\,b-4\,b^2-a)\\
R_4^3 &= \frac{1}{\Delta}(-fkb^2-2\,fkb-fk+4\,m)\\
R_1^4 &= 0, R_2^4=\frac{ak}{\Delta}, R_3^4=\frac{am}{\Delta}\\
R_4^4 &= \frac{1}{\Delta}(3\,f^2b^2+3\,f^2+6\,f^2b-4-4\,b-4\,b^2).
\end{aligned}$$

3.11 Algebra $4.10(a=0)$ $4.11a(a>0)$

$4.10/4.11a(a>0)$ $[e_2,e_3]=e_1, [e_1,e_4]=2ae_1, [e_2,e_4]=ae_2-e_3, [e_3,e_4]=e_2+ae_3$

$$S=\begin{bmatrix} e^{2aw} & -e^{aw}(x\sin w+y\cos w) & e^{aw}(x\cos w-y\sin w) & z\\ 0 & e^{aw}\cos w & e^{aw}\sin w & x\\ 0 & -e^{aw}\sin w & e^{aw}\cos w & y\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Left-invariant vector fields $2e^{2aw}D_z, e^{aw}(\cos wD_x - \sin wD_y - (y\cos w + x\sin w)D_z)$, $e^{aw}(\sin wD_x + \cos wD_y + (x\cos w - y\sin w)D_z), -D_w$

Left-invariant one forms $e^{-2aw}(dz+ydx-xdy), e^{-aw}(\cos wdx-\sin wdy), e^{-aw}(\sin wdx+\cos wdy), dw$

Right-invariant vector fields $-2D_z, D_x + yD_z, D_y - xD_z, D_w + (ax + y)D_x + (ay - x)D_y + 2azD_z$

Right-invariant one forms $dz - ydx + xdy + (x^2 + y^2 - 2az)dw, dx - (ax + y)dw, dy + (x - ay)dw, dw$.

The space of derivations is given by $\begin{bmatrix} 2\alpha & \gamma & \delta & \epsilon \\ 0 & \alpha & \beta & a\delta-\gamma \\ 0 & -\beta & \alpha & -a\gamma-\delta \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & e & 0 & k \\ 0 & 0 & 1 & m \\ 0 & k & m & 1 \end{bmatrix} \tag{3.65}$$

where $b > 0, e > 0, e - em^2 - k^2 > 0$.

3.11.1 Ricci Tensor

The R(0,2) tensor has the following components;

$$\begin{aligned}
R_{11} &= \frac{1}{\Delta}(-b\left(16\,a^2e - b\right)) \\
R_{12} &= \frac{1}{\Delta}(be\left(5\,am + k\right)) \\
R_{13} &= \frac{1}{\Delta}(-b\left(-m + 5\,ak\right)) \\
R_{14} &= \frac{1}{\Delta}(b\left(m^2 + k^2\right)) \\
R_{22} &= \frac{1}{\Delta}(-eb + bk^2 - 8\,e^2a^2 + e^3 - e) \\
R_{23} &= \frac{1}{\Delta}(mkb + 4\,ae - 4\,ae^2) \\
R_{24} &= \frac{1}{\Delta}(4\,eam - 4\,e^2am - 8\,ka^2e + ke^2 - k) \\
R_{33} &= \frac{1}{\Delta}(-b + bm^2 - 8\,a^2e - e^2 + 1) \\
R_{34} &= \frac{1}{\Delta}(4\,ak - 4\,kea - 8\,a^2em - me^2 + m) \\
R_{44} &= \frac{1}{\Delta}(4\,a^2em^2 - 12\,a^2e + 4\,k^2a^2 + 8\,akm - 8\,amek - e^2 - 2\,em^2 + 2\,e + 2\,m^2 - 1 + 2\,k^2e - 2\,k^2)
\end{aligned}$$

and

$$R_{21} = R_{12},\ R_{31} = R_{13},\ R_{41} = R_{14},\ R_{32} = R_{23},\ R_{42} = R_{24},\ R_{43} = R_{34},\ \Delta = 2(e - em^2 - k^2).$$

The Ricci(1,1) form is given by

$$R = \frac{1}{\Delta}\begin{bmatrix} -16\,a^2e + b & b\,(5\,am + k) & -b\,(-m + 5\,ak) & 0 \\ e\,(5\,am + k) & e^2 - 8\,a^2e - b - 1 & -4\,e\,(-1 + e)\,a & kb \\ m - 5\,ak & -4\,a\,(-1 + e) & -e^2 - 8\,a^2e - b + 1 & mb \\ m^2 + k^2 & 4am - 4\,eam + 2\,ke + 4\,ka^2 - 2\,k & 4\,a^2em - 2\,me - 4\,kea + 2\,m + 4\,ak & 2e - e^2 - 12\,a^2e - 1 \end{bmatrix}.$$

3.12 Algebra 4.12

$4.12[e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1$

$$S = \begin{bmatrix} e^z \cos w & e^z \sin w & x \\ -e^z \sin w & e^z \cos w & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Left-invariant vector fields: $e^z(\cos wD_x - \sin wD_y), e^z(\sin wD_x + \cos wD_y), D_z, D_w$

Left-invariant one forms: $e^{-z}(\cos wdx - \sin wdy), e^{-z}(\sin wdx + \cos wdy), dz, dw$

Right-invariant vector fields: $D_x, D_y, D_z + xD_x + yD_y, D_w + yD_x - xD_y$

Right-invariant one forms: $dx - xdz - ydw, dy - ydz + xdw, dz, dw$.

3.12.1 Reduction of metric

The space of derivations is given by $der(\mathfrak{g}) = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & \delta & -\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the reduced metric by

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & b & 1 & d \\ 0 & c & d & e \end{bmatrix} \tag{3.66}$$

where $a > 0, a - b^2 > 0, ae - ad^2 - b^2e + 2bcd - c^2 > 0$.

3.12.2 Ricci Tensor

The R(0,2) tensor has following components:

$$
\begin{aligned}
R_{11} &= \frac{1}{\Delta}(-a^2 - 4\,ae + ab^2 + 1 + 4\,c^2) \\
R_{12} &= \frac{1}{\Delta}(-2\,da^2 + 2\,da + 2\,bca - 3\,bc) \\
R_{13} &= \frac{1}{\Delta}(ca - 2\,bda + cb^2 - bd) \\
R_{14} &= \frac{1}{\Delta}(-dac + 2\,cd - 3\,be + bc^2) \\
R_{22} &= \frac{1}{\Delta}(-4\,a^2e + 5\,c^2a + a^3 - a - b^2a^2 + b^2) \\
R_{23} &= \frac{1}{\Delta}(-4\,bae + 3\,dac + 2\,bc^2 + ba^2 - ab^3) \\
R_{24} &= \frac{1}{\Delta}(bd + ca^2 - c - cae + 2\,c^3 - cb^2a) \\
R_{33} &= \frac{1}{\Delta}(-4\,ae + 4\,ad^2 - 2\,bcd + 3\,c^2 + ab^2 - b^4) \\
R_{34} &= \frac{1}{\Delta}c\left(ba - be + 2\,cd - b^3\right) \\
R_{44} &= \frac{1}{\Delta}(-e - 2\,bdac + aeb^2 - a^2e + 2\,c^2a - c^2b^2 + d^2a^2 + d^2 + 2\,ae - 2\,ad^2 - 2\,b^2e - 2\,c^2 + 4\,bcd + ec^2)
\end{aligned}
$$

and

$$R_{21} = R_{12},\ R_{31} = R_{13},\ R_{41} = R_{14},\ R_{32} = R_{23},\ R_{42} = R_{24},\ R_{43} = R_{34},\ \Delta = 2(ae - ad^2 - b^2e + 2bcd - c^2).$$

The Ricci(1,1) form is given by $R =$

$$\frac{1}{\Delta}\begin{bmatrix} -a^2 - 4\,ae + ab^2 + 1 + 4\,c^2 & -2\,da + 2\,d + bc & ca & -3\,b \\ -2\,da^2 + 2\,da + 2\,bca - 3\,bc & a^2 - ab^2 - 4\,ae - 1 + 2\,c^2 & b & 3\,ca \\ ca - 2\,bda + cb^2 - bd & ba - cd - b^3 & -4\,ae + 3\,c^2 & 4\,da - bc \\ -dac + 2\,cd - 3\,be + bc^2 & -bda + 2\,ca + 2\,bd - cb^2 - 2\,c - ce & da^2 - 2\,da - bca + d + 2\,bc & -a^2 + 2\,a + ab^2 - 1 - 2\,b^2 + 2\,c^2 \end{bmatrix}.$$

3.13 Einstein Spaces

In this Section we take up the question of finding those four-dimensional Lie groups that correspond to Einstein spaces. We can say immediately in view of Theorem (2.1.1) that of the Algebras 4.1-4.6, only Algebras 4.5($a = b = 1$) and 4.6 ($a = b \neq 0$) can be associated to an Einstein space and they are both of constant curvature. In this Section the Ricci tensor will always mean the type (1,1) Ricci tensor.

3.13.1 Algebra 4.7

The $(1,4)$-entry in the Ricci tensor is given by $R_4^1 = -\frac{b^2}{b^2+c^2e-ae}$ and hence to obtain an Einstein space we must have $b=0$. Now the Ricci (1,1)-tensor is given by

$$R = \frac{-1}{2e(a-c^2)} \begin{bmatrix} a^2-16\,ae+12\,c^2e & 3\,e^2c & \left(2\,a+e^2-3\,e\right)c & 0 \\ -eca & ae^2-a^2-8\,ae+8\,c^2e & e\left(3\,c^2-4a\right) & -ca \\ 5\,eca & -4\,e^2a & 12c^2e-a^2-e^2a-8\,ae & 0 \\ 0 & -eca & 0 & e\left(11c^2-12\,a-ae\right) \end{bmatrix} \tag{3.67}$$

Since $a>0$ looking at R_2^3 we find that $e=0$ which implies that the matrix g is not negative definite. Hence there is no Einstein metric.

3.13.2 Algebra 4.8/4.9b

Since $a>0$ we must have that $k=m=0$ in order to obtain an Einstein space. The Ricci tensor R is given by $-2(1-f^2)R$=

$$\begin{bmatrix} 4(b^2f^2+2f^2b+f^2+\frac{a}{4}-2b-1-b^2 & 0 & 0 & 0 \\ 0 & 2\,b^2f^2+4\,f^2b+2\,f^2-4\,b-4-a & (3\,b+1)(b-1)f & 0 \\ 0 & -(b+3)(b-1)f & 2\,b^2f^2+4\,f^2b+2\,f^2-a-4\,b^2-4\,b & 0 \\ 0 & 0 & 0 & 3\,b^2f^2-4\,b^2-4\,b+6\,f^2b-4+3\,f^2 \end{bmatrix} \tag{3.68}$$

Looking at R_3^2 and R_2^3 we conclude that $(b-1)f=0$.

Now we consider two cases supposing first of all that $b=1$. Then comparing R_1^1 and R_2^2 we find that $a=4(1-f^2)$. At this point we have an Einstein space with $\rho=-24$ which is not of constant curvature. In the other case we assume that $f=0$. Then comparing the $(3,3)$ and $(4,4)$-entries we find that $a=4$. Then comparing R_1^1 and R_3^3 we find that $b=1$. However, this latter case is comprised in the case already found above.

In the case where $b=1$ the space of derivations is actually given by

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \epsilon & \phi & \gamma \\ 0 & \psi & \alpha-\epsilon & -\beta \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.69}$$

so that the parameter f in the metric g may be removed.

3.13.3 Algebra 4.10/4.11a

The $(1,4)$ entry in the Ricci tensor is given by

$$R_4^1 = -\frac{k^2+m^2}{2(e-em^2-k^2)}.$$

Hence we must have that $k=m=0$ in order to obtain an Einstein space. Then, the Ricci tensor is then given by

$$R = \frac{-1}{2e}\begin{bmatrix} b-16\,a^2e & 0 & 0 & 0 \\ 0 & e^2-8\,ea^2-b-1 & 4\,a(1-e) & 0 \\ 0 & 4\,e(1-e)a & 1-e^2-8\,ea^2-b & 0 \\ 0 & 0 & 0 & 2e-12\,ea^2-e^2-1 \end{bmatrix}. \tag{3.70}$$

Now comparing R_2^2 and R_3^3 we find that $e=1$ since $e>0$ and from R_1^1 and R_2^2 that $b=4a^2$. At this point we have an Einstein space with $\rho=-24a^2$ which is not a space of constant curvature.

3.13.4 Algebra 4.12

The $(4,2)$ and $(4,3)$-entries in the Ricci tensor are given by

$$R_2^4 = \frac{c}{2(ae-ad^2-b^2e+2bcd-c^2)},\ R_3^4 = \frac{d}{2(ae-ad^2-b^2e+2bcd-c^2)}.$$

Hence we must have that $c = d = 0$ in order to obtain an Einstein space. The Ricci tensor is now given by

$$R = \frac{1}{2e(a-b^2)} \begin{bmatrix} -a^2+ab^2-4\,ae+1 & 0 & 0 & -3\,be \\ 0 & a^2-4\,ae-ab^2-1 & b\left(a-b^2\right) & 0 \\ 0 & b & -4\,ae & 0 \\ -3\,b & 0 & 0 & 2\,a-2\,b^2-a^2-1+ab^2 \end{bmatrix}.$$

Looking at R^4_1 we must have that $b = 0$. Now equating R^1_1 and R^2_2 we find that $a^2 = 1$ and since $a > 0$ that $a = 1$. However, we find now that $R^4_4 = 0$ whereas $R^3_3 < 0$ and the space is not Einstein.

3.14 Decomposable Algebras

In this Section we consider the four-dimensional decomposable Lie groups. Again the three-dimensional *indecomposable* Lie algebras can be found in [PSWZ].

3.14.1 Algebra $3.1 \oplus \mathbb{R}$

Brackets: $[e_2, e_3] = e_1$. Algebra 3.1 is the Heisenberg algebra, but its product with $\mathbb{R}$ is a nilpotent Lie algebra and hence according to [Mil] cannot be associated to an Einstein space as we found for Algebra 4.1.

3.14.2 Algebra $3.2 \oplus \mathbb{R}$

Brackets: $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$. The space of derivations is given by $\begin{bmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \alpha & \delta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & \phi \end{bmatrix}$ and the metric can be reduced to

$$g = \begin{bmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ b & c & 0 & 1 \end{bmatrix}. \tag{3.71}$$

Ricci is given by

$$2(a-b^2-ac^2)R = \tag{3.72}$$

$$\begin{bmatrix} a^2+4\,ac^2-bca-ab^2-4\,a+2\,b^2 & 3\,ac^2-bca-2\,a-2\,bc+b^2 & 0 & ab+ac-b^2c-b-b^3 \\ -2\,a\left(bc+a-b^2\right) & ab^2-a^2+2\,ac^2-bca-4\,a+4\,b^2 & 0 & -ab-ac+b^3 \\ 0 & 0 & ab^2-a^2+3\,ac^2-2\,bca-4\,a+3\,b^2 & 0 \\ 3\,a(ac+b) & a(ac+3\,c+b) & 0 & 2\,b^2+2\,bca+2\,ac^2 \end{bmatrix}.$$

In order to obtain an Einstein space looking at R^4_1 and R^4_2 we must have that $b = c = 0$ since $a > 0$. However, now $R^1_2 = -1$ and the space is not Einstein.

3.14.3 Algebra $3.3(a = 1), 3.4(a = -1), 3.5(0 < |a| < 1) \oplus \mathbb{R}$

Brackets: $[e_1, e_3] = e_1, [e_2, e_3] = ae_2$. The space of derivations is given by $\begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & \phi \end{bmatrix}$ and the metric can be reduced to

$$g = \begin{bmatrix} 1 & b & 0 & c \\ b & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}. \tag{3.73}$$

We find that $R^1_2 = \frac{a(ba-b+bd^2-dca)}{(-b^2-c^2-d^2+b^2+2bcd)}, R^1_4 = \frac{a(c-cd^2-abd+acd^2)}{2(1-b^2-c^2-d^2+2bcd)}$. Regarding $R^1_2 = 0, R^1_4 = 0$ as a system of homogeneous linear equations for b and c we find that the determiniant of the matrix of coefficients is $-(a-1)(d-1)^2(d+1)^2$. Hence either $a = 1$ or else $b = c = 0$ since $d^2 < 1$. Suppose the latter case holds. Then Ricci is given by

$$R = \frac{1}{2(1-d^2)} \begin{bmatrix} 2\left(-1+d^2\right)(a+1) & 0 & 0 & 0 \\ 0 & \left(-2+d^2\right)(a+1)a & 0 & -da \\ 0 & 0 & 2\,d^2-2+d^2a^2-2\,a^2 & 0 \\ 0 & (1+2\,a)da & 0 & (a+1)d^2a \end{bmatrix}. \tag{3.74}$$

From R^2_4 we conclude that $d = 0$ but now $R^4_4 = 0$ whereas $R^3_3 < 0$ and hence the space is not Einstein.

In the other case we assume that $a = 1$. This time we find that

$$R_1^4 = \frac{3(1-b^2)c}{2(1-b^2-c^2-d^2+2bcd)}, R_2^4 = \frac{3(1-b^2)d}{2(1-b^2-c^2-d^2+2bcd)}.$$

Since $1-b^2 > 0$ we must have $c = d = 0$. Now we find that $R_4^4 = 0$ whereas $R_3^3 = -2$ and again the space is not Einstein.

3.14.4 Algebra $3.6(a = 0), 3.7(a > 0) \oplus \mathbb{R}$

Brackets: $[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2$. The space of derivations is given by $\begin{bmatrix} \alpha & \beta & \gamma & 0 \\ -\beta & \alpha & \delta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & \phi \end{bmatrix}$ and the metric can be reduced to

$$g = \begin{bmatrix} 1 & 0 & 0 & c \\ 0 & b & 0 & d \\ 0 & 0 & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}. \tag{3.75}$$

We find that $R_1^4 = \frac{(b^2c+3a^2bc-abd+3ad)}{2(b-d^2-bc^2)}, R_2^4 = \frac{(abc-a3b^2c+3a^2bd+d)}{2(b-d^2-bc^2)}$. Regarding $R_2^1 = 0, R_4^1 = 0$ as a system of homogeneous linear equations for c and d we find that the determinant of the matrix of coefficients is $(1+a^2)(9a^2+1)b^2$. Since $b > 0$ we deduce that $c = d = 0$ and then $R_2^1 = a(b-1)$ and $R_4^4 = 0$. Since $b^2 < 1$ we must have that $a = 0$. Now provided $b = 1$ we find that $R = 0$. However, this metric is the product of a flat metric on Algebra 3.6 and a flat de Rham factor and therefore is not Einstein.

3.14.5 Algebra $3.8 \oplus \mathbb{R}$

Brackets: $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ The space of derivations is given by $\begin{bmatrix} 0 & \gamma & \beta & 0 \\ -\gamma & 0 & \alpha & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix}$ and the metric can be reduced to

$$g = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & 0 & e \\ 0 & 0 & c & f \\ d & e & f & 1 \end{bmatrix}. \tag{3.76}$$

We find that

$$R^4_1 = \frac{d(b+c)^2}{2(abc - abf^2 - ace^2 - bcd^2)},\ R^4_2 = \frac{e(a+c)^2}{2(abc - abf^2 - ace^2 - bcd^2)}$$

$$R^4_3 = \frac{f(a-b)^2}{2(abc - abf^2 - ace^2 - bcd^2)}, R^4_4 = 0.$$

Since a, b, c are positive we must have that $d = e = 0$ in order to have an Einstein space. Now we find that Ricci is given by

$$R = \frac{1}{2ab(c - f^2)} \begin{bmatrix} a^2-b^2-c^2+2f^2b+f^2c-2bc & 0 & 0 & 0 \\ 0 & b^2-a^2-c^2+2f^2a+f^2c-2ac & 0 & 0 \\ 0 & 0 & c^2-a^2-b^2+2ab-f^2c & f\left(c-f^2\right) \\ 0 & 0 & f(a-b)^2 & 0 \end{bmatrix}. \tag{3.77}$$

Now $R^3_4 = 0$ implies that $f = 0$.

Thus if $ef \neq 0$ then $a = b = c$ and then $R^1_1 = \frac{1}{2a}$ and the space is not Einstein and likewise if $de \neq 0$ and $df \neq 0$. Hence we may assume that $e = f = 0$. Then $R^1_4 = \frac{d}{2bc}$ and hence $d = 0$. Now we find that Ricci is given by

$$R = \begin{bmatrix} \frac{(a-b+c)(a+b-c)}{2abc} & 0 & 0 & 0 \\ 0 & \frac{(b-c+a)(b+c-a)}{2abc} & 0 & 0 \\ 0 & 0 & \frac{(c-a+b)(c+a-b)}{2abc} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.78}$$

Now to obtain an Einstein space we need either $a = b + c$ or $b = a + c$ but in the former case $R_2^2 = -\frac{2}{b}$ and in the latter $R_1^1 = -\frac{2}{a}$ which contradicts $R_4^4 = 0$.

3.14.6 Algebra $3.9 \oplus \mathbb{R}$

Brackets: $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. The space of derivations is given by $\begin{bmatrix} 0 & \gamma & -\beta & 0 \\ -\gamma & 0 & \alpha & 0 \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix}$ and the metric can be reduced to

$$g = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & 0 & e \\ 0 & 0 & c & f \\ d & e & f & 1 \end{bmatrix}. \tag{3.79}$$

We find that

$$R_1^4 = \frac{d(b-c)^2}{2(abc - abf^2 - ace^2 - bcd^2)}, \; R_2^4 = \frac{e(c-a)^2}{2(abc - abf^2 - ace^2 - bcd^2)}$$

$$R_3^4 = \frac{f(a-b)^2}{2(abc - abf^2 - ace^2 - bcd^2)}, R_4^4 = 0.$$

Thus if $ef \neq 0$ then $a = b = c$ and then $R_1^1 = \frac{1}{2a}$ and the space is not Einstein and likewise if $de \neq 0$ and $df \neq 0$. Hence we may assume that $e = f = 0$. Then $R_4^1 = \frac{d}{2bc}$ and hence $d = 0$. Now we find that Ricci is given by

$$R = \begin{bmatrix} \frac{(a-b+c)(a+b-c)}{2abc} & 0 & 0 & 0 \\ 0 & \frac{(b-c+a)(b+c-a)}{2abc} & 0 & 0 \\ 0 & 0 & \frac{(c-a+b)(c+a-b)}{2abc} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.80}$$

From here we find easily that we cannot have an Einstein space: for example $c = a+b$ gives $R_3^3 = \frac{2}{a+b}$ whereas $R_4^4 = 0$.

3.14.7 Algebra $2.1 \oplus 2.1$

Brackets: $[e_1, e_2] = e_2, [e_3, e_4] = e_4$. In this case there are no outer derivations. When the metric is reduced it depends on five parameters and it is found to be Einstein precisely when it is of the form

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.81}$$

where $a > 0$. The metric is a product of two spaces of constant curvature -1. The $(3, 3)$-entry could in fact be any positive number but it can be reduced to unity by scaling.

3.14.8 Algebra $2.1 \oplus \mathbb{R}^2$

Brackets: $[e_1, e_2] = e_2$. The space of derivations is of dimension eight and the reduced metric depends on two parameters. It easily follows that there is no Einstein metric.

3.14.9 Abelian Algebra $\mathbb{R}^4$

No non-zero brackets. Every invariant metric is flat and can be reduced to the standard flat Pythagorean metric.

3.15 Summary

The following table gives a list of the Einstein metrics in dimension four.

Theorem 3.15.1. *Every Einstein Riemannian metric on a four dimensional Lie group is one of the following seven types five of which are constant curvature spaces* *and denoted by* *.

Algebra	*Einstein and/or (Constant Curvature*) Metric*
$4.5(a=b=1)^*$	$(dx-xdw)^2+(dy-ydw)^2+(dz-zdw)^2+dw^2$
$4.6(a=b=1)^*$	$(dx-axdw)^2+(dy-(by+z)dw)^2+(dz-(bz-y)dw)^2+dw^2$
$4.9(b=1)$	$4(dx-zdy+(yz-2x)dw)^2+(dy-ydw)^2+(dz-zdw)^2+dw^2$
$4.11a(a>0)$	$4a^2(dz-ydx+xdy+(x^2+y^2-2az)dw)^2+(dx-(ax+y)dw)^2+(dy+(x-ay)dw)^2+dw^2$
$3.6\oplus\mathbb{R}^*$	$(dx-ydz)^2+(dy+xdz)^2+dz^2+dw^2$
$2.1\oplus 2.1^*$	$dx^2+(dy-ydx)^2+dz^2+a(dw-wdz)^2.$
$\mathbb{R}^{4*}$	$dx^2+dy^2+dz^2+dw^2.$

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