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A Leibnizian approach to mathematical relationships : a new look at synthetic judgments in mathematics

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A Thesis

entitled

A Leibnizian Approach to Mathematical Relationships: A New Look at
Synthetic Judgments in Mathematics

by

David T. Purser

Submitted to the Graduate Faculty as partial fulfillment of the requirements for the
Master of Arts Degree in Philosophy

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An Abstract of
A Leibnizian Approach to Mathematical Relationships: A New Look at Synthetic
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I examine the methods of Georg Cantor and Kurt Gödel in order to understand how new symbolic innovations aided in mathematical discoveries during the early 20th Century by looking at the distinction between the *lingua characteristica* and the *calculus ratiocinator* in the work of Leibniz. I explore the dynamics of innovative symbolic systems and how arbitrary systems of signification reveal real relationships in possible worlds. Examining the historical articulation of the analytic/synthetic distinction, I argue that mathematics is synthetic in nature. I formulate a moderate version of mathematical realism called modal relationalism.

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Preface

Mathematics and philosophy have a complex intertwined history. Prominent figures in the history of both disciplines are the very same people, especially in the modern period: Descartes, Pascal and Leibniz, to name a few. The contributions of these scholars transformed both of these disciplines significantly. This thesis examines the scholarly writings of Leibniz in an effort to determine the legacy of some of his unique and original proposals regarding the intersection of metaphysics and mathematics. In particular, I consider his *lingua characteristica* and *calculus ratiocinator* and the extent to which these ideas influenced the development of philosophical perspectives on the relationship between formal languages and reality.

The first chapter is a historical approach to the relationship between logic and language within the work of Leibniz. I will discuss Leibniz's vision for a universal language or *lingua characteristica* and a logical calculus or *calculus ratiocinator*. What are the similarities and differences between these proposals and to what extent were either or both of these visions realized? The conclusion of the chapter will consider the impact that Leibniz had on early 20th Century philosophy.

The second chapter considers Cantor, Gödel and the paradoxes in mathematics that arose in the early 20th C. The chapter makes Leibnizean connections between Cantor's diagonal method, Russell's Paradox and Gödel's Incompleteness theorem. The main focus will be on Gödel's second theorem because Gödel explicitly refers to

Leibniz's project and sees his own work as fulfilling Leibniz's vision, though the extent to why this is the case will be introduced as an important question.

Chapter three places this particular historical question; did Gödel realize Leibniz's vision, in a larger metaphysical context: What do these paradoxes tell us about the nature of mathematics and logic? Is there any connection between Leibniz's thought and these paradoxes? To answer this question I discuss the distinction between a Kantian understanding of arithmetic as synthetic *a priori* and the Logical Positivists understanding of arithmetic as analytic *a priori*. Is arithmetic no more than mere tautology? And if it is, does this imply that mathematics is void of content? Is there any formal system with necessarily true propositions that also contains content? Ultimately, asking these questions require that we consider whether the discovery of these paradoxes have inherent implications for philosophy of mathematics. I conclude by examining the distinction between realism and nominalism in the philosophy of mathematics. I explore how Leibniz's distinctions and the formulation of formal paradoxes in the early 20th C. shed light on the realism debate.

Chapter 1

Leibniz and Symbolic Language

1.1 Introduction

Leibniz's scholarly work is traditionally classified into two categories, mathematics and philosophy. On the one hand, Leibniz is well-known for his invention of the calculus and the *Theodicy*. On the other hand, Leibniz worked his entire life on a universal language, or the *lingua characteristic* (Universal Characteristic), and on *calculus ratiocinator*, (Logical Calculus). These two ideas permeate his work from his dissertation until the end of his life. In order to understand Leibniz's metaphysics and mathematics, we need to understand how these ideas fit into Leibniz's lifelong work. I examine the "Dissertation on the Art of Combinations" (1666), "Letter to Thomas Hobbes" (1670), "Letter to Duke Johann Friedrich" (1671), "Compendium of the calculus of the direct method of tangents, together with its use for finding tangents to other curves. Also some observations on the inverse method" (1675), "On the General Characteristic" (1679), "Meditations on Knowledge, Truth and Ideas" (1684), "*Generales Inquisitiones De Analyysi Notionum Et Veritatum*" (1686a), "Discourse on Metaphysics" (1686b), "Necessary and Contingent Truth" (1686c), and the "Letter to Gabriel Wagner on the Value of Logic" (1696).¹

¹ Translated out of Loemker (1969) unless otherwise noted.

The Universal Characteristic was to reflect the structure of human concepts in a universal way. It would be a language of universal thought (Hintikka 1997, ix). In contrast, the Logical Calculus was to provide the method for using the Characteristic in order to adequately represent the reasoning of humans (Hintikka ix). In this chapter, I will discuss this strand of thought in the writing of Leibniz, in order to show that well-chosen symbols have the power to reveal meaningful relationships about the real world. First, I will analyze Leibniz' vision for a Universal Characteristic, showing that arbitrary symbols systems make evident real relationships that we could not otherwise see by eliminating the dangers of ambiguity and equivocation. Secondly, I will examine Leibniz' work on a Logical Calculus, the process of reasoning within an arbitrary symbol system and how this reveals new relationships in the world. In conclusion, I will bring these two ideas together to show that Leibniz is a precursor for the invention of our current system of symbolic logic. I also examine the legacy that Leibniz left behind by exploring Leibniz's theory of relations as both mathematically representable and metaphysically real.

Leibniz was criticized because of his views on the usefulness of logic. In correspondence with Gabriel Wagner, Leibniz presents the value of a symbolic logic. Leibniz (1696) says, "All our logics until now are but a shadow of what I should wish and what I see from afar" (463). Leibniz sees that Aristotelian and Scholastic logical systems are useful but not as valuable as they could be.

In order for a new symbol system to be of more value than classical systems, it must exhibit the same rigor as mathematics. Leibniz (1696) says, "But I consider it certain that the art of reasoning can be carried incomparably higher and believe not only

that I see this but that I already have a foretaste of it, which I could hardly have attained, however, without mathematics” (467). Leibniz predicts that the connection between logic and mathematics allows a more valuable logical system. If we hold all of our science to the same rigorous standards as mathematics then Leibniz believes that we would be able to make great progress in our understanding of science. Not only do the scientists make progress in thought, but the human condition is improved. Leibniz (1696) says, “So I am convinced that with the advantage of these aids and the willingness to use them, a poor head could excel the best, just as a child with a ruler can draw better lines than the greatest master with free hand” (465). All people would be able to use logic to sharpen their minds. After learning the rules that govern the relationships that are inherent in the world, one is more inept to learn new things than one who tries to learn without knowing the necessary rules of thought.

Why would Leibniz spend so much time working on a Universal Characteristic and a Logical Calculus? Leibniz believed that these two things working together would help not only the scientific community make progress in discovering truth but also would help all of humanity by sharpening the minds of all people, not just the elite of society. The need for a universal, scientific language is so that people from all over the world could come and reason together. This allows people to understand and work out problems together with the guarantee of reaching the same conclusions.

Leibniz’s vision wherein the discovery of truth is a collaborative effort should be remembered as we consider the idea of the Universal Characteristic and Logical Calculus for two reasons. First, the idea of a Universal Characteristic that covers every concept appears to be a utopian dream (Peckhaus 2004, 8). The Universal Characteristic, for

Leibniz, was never supposed to be something that he attempted to do on his own, but required help from a co-operative group (Parkinson 1966, xviii). Second, this logical project is not meant to serve the elite of society in isolation, but was meant to be a tool that could be used by all in order to discover new truth. Thus, when Leibniz uses the term ‘universal’ he does not just mean an abstract, context-free language, but a language that is accessible to all.

1.2 Universal Characteristic

The Universal Characteristic was a life-long project of Leibniz. In speaking of the Universal Characteristic, Donald Rutherford (1995) said, “Leibniz considered his plan for a universal characteristic to be among the most important of his inventions; and, particularly in his early years, he endowed it with extraordinary powers” (231). If the Universal Characteristic is the most important invention of Leibniz then we need to examine this in detail, in order to see how the universal characteristic helps us understand the world.

The idea of the Universal Characteristic is that humans, not just Leibniz, could produce a universal language that would bring out all the logical connections between the different components of language. Rutherford says, “The universal characteristic would enable us to construct linguistic characters which are transparent representation of intelligible thoughts” (225). A formal language that explicitly brings out the logical relationships between propositions would have advantages over natural languages such as avoiding equivocations and ambiguity. The language is also more advantageous because it allows the implicitly logical relationships to become evident to our senses. When I say

evident to our senses, I do not mean that we literally see the logical relationships but that the clearer and more precise the symbol of signification the easier the complex relationships between objects or concepts becomes to comprehend.

The Universal Characteristic is composed of two different parts. The first component is the reduction of all concepts to their simplest forms. The second component of the Universal Characteristic is the arbitrary nature of the signs that stand for all of the concepts. In order to understand what Leibniz's dream of a Universal Characteristic teaches us, it is necessary to examine both components.

1.2.1 Simple Concepts

Leibniz believed that all concepts are reducible to simple terms. Deriving the simple terms for every concept is essential to the formulation of the Characteristic. In discussing the idea of the Characteristic, Leibniz (1671) said,

By its means all notions in the world can be reduced to a few simple terms which serve as an alphabet and by using combinations of this alphabet a means will be found to discover systematically all things together with their theorems and whatever can be found out about them. This invention, in so far as God willing it may be put to work, is considered by me to be the mother of all inventions and to be the most important of all, even though it may not be recognized as such at the moment. I have, by its means, discovered everything that is now to be related and I hope to discover even more (Martin 1960, 63-64).

In order to achieve the Characteristic one must create an alphabet that reduces concepts to a few simple terms. This alphabet is created, and thus arbitrary.

Leibniz began working on the Universal Characteristic early in his career. His 1666 dissertation contains the plan for the Characteristic (73). In his dissertation, Leibniz claims that every relation is one of harmony, which can be divided into different parts

(76). It is here that we see the beginning of the thought that all concepts or relations can be broken into simple components.

Leibniz in 1679 discussed the idea of the Universal Characteristic in detail. He said that the Universal Characteristic is, “A kind of alphabet of human thoughts can be worked out and that everything can be discovered and judged by a comparison of the letters of this alphabet and an analysis of the words made from them” (222). This alphabet involves reducing all of our concepts to not only simple terms but in reducing our concepts to numbers (1679, 224). Every letter in our formal alphabet will correspond to a specific number. Here we see an anticipation of, if not a forerunner of Gödel numbering. Numbers are clear and distinct and therefore numbers allow us to avoid ambiguity/equivocation. This goal to reduce all concepts to the simplest terms appears to be a utopian dream that is unachievable.

It is easy to dismiss Leibniz as a wishful thinker but Leibniz himself was not so naïve. Leibniz said that, “Even though such a language depends upon a true philosophy, it does not depend upon its perfection” (Eco 1995, 277, translated from Couturat 1903, 27-28). Leibniz argued that the language could still be constructed and employed to good effect even if it was not perfect. According to Leibniz, as our philosophy and science improve so will our language. Even if our language is imperfect, the language could still be useful. Leibniz goes on to say that, “In the meantime, it will continue to perform an admirable service by helping us retain what we know, showing us what we lack, and inventing means to fill that lack. Most of all, it will serve to avoid those disputes in the science that are based on argumentation” (Eco 1995, 277, translated from Couturat 1903, 27-28).

The Universal Characteristic provides a useful way to situate simple terms in an analyzable language. If the language never becomes perfect, it will still be useful because of the power, which a rigorous language would provide. That power includes the ability to solve disputes in arguments by a calculation. The rigorous language also provides a means of avoiding disputes. Parkinson (1966) says that in knowing this alphabet we would have a powerful instrument of deductive proof (xv). Not only would it be a powerful instrument, but Leibniz would argue that it is the most powerful scientific instrument that we have at our disposal. Leibniz (1679) said that, “Once the characteristic numbers for most concepts have been set up, however, the human race will have a new kind of instrument which will increase the power of mind much more than optical lenses strengthen the eyes and which will be as far superior to microscopes or telescopes as reason is superior to sight” (224). In other words, the Universal Characteristic provides a scientific revolution that goes beyond pure science to every realm of life.

An analyzable language is desirable for two different reasons. First, an analyzable language allows for rigor in our language. A rigorous language will allow us to see exactly where inconsistencies appear in our moral, metaphysical and scientific knowledge. Second, an analyzable language allows us to see the strengths and weaknesses of competing hypotheses more clearly, or for the first time. It might be the case that neither of our hypotheses are perfect, but the analyzable language allows us to see exactly where the fault is in our reasoning, and what steps could correct the fault in order to provide a stronger hypothesis.

1.2.2 Arbitrary Signs

How are we to create this alphabet of simple concepts? This question was very important for Leibniz. He believed that in order to complete this alphabet that it was necessary to have signs that represent the concepts. Signs are necessary in this alphabet because of the role of signs in all of our language and not just in technical discourse. Martin (1964) argues that, “We can say that in mathematics and physics, the sciences in which Leibniz was interested, signs are a necessity. Leibniz then seems to have gone on to argue that signs were necessary for all kinds of thought” (60).

Leibniz argues that signs are necessary in *Meditations on Knowledge, Truth and Ideas* (1684). Here, Leibniz argues that we use signs in all of our thinking by using an example from mathematics. He explains that when solving a mathematical problem we do not think of the entire problem at once, but that we use signs to represent the problem to make things easier for us. He continues his example by saying:

Thus when I think of a chiliogon, or a polygon of a thousand equal sides, I do not always consider the nature of a side and of equality and of a thousand, but I use these words...in place of the ideas which I have of them...Such thinking I usually call *blind* or *symbolic*; we use it in algebra and in arithmetic, and indeed almost everywhere. When a concept is very complex, we certainly cannot think simultaneously of all the concepts which compose it (292).

Conceptual complexity is not confined to mathematics. Everyday discourse involves complex concepts and relations. Because of the complexity of many of the concepts that we use on a daily basis we do not always take into consideration what every word implies. For instance, when I tell a friend that last night the Braves beat the Mets 5 to 2 I might think of the home run that was hit in the fifth inning, but I do not think of all 54

outs. Leibniz understood that familiarity to concepts can blind us to the complexity inherent to our everyday discourse.

What kinds of signs are most effective? The early work of Leibniz (1679) argues that the signs that have the properties we want are numerals. The reason that numerals are useful is because of, “the accuracy and ease with which they are handled” (235). Leibniz knows that working with basic arithmetic is fairly simple and leads to the same kind of certainty which his logical method will provide for other contexts. In other words, Leibniz argues that using numbers, as our arbitrary parsigraphic symbols, allows one to see the certainty of the proofs easier than using any other form of language.

Leibniz realizes that the arbitrary signs which we choose do not matter in the end. The choice of signs does not matter because the important aspect of this language is the relationships between concepts which remains the same no matter what the signs are (1679, 241). Therefore, if we were to change the signs, and do this consistently, it would not change the meaning of our calculations. This implies that it is not the concepts or objects in and of themselves that are important but how they relate to one another. Thus, Leibniz (1679) says, “To understand the nature of this calculus, we must note that whatever we express by certain letters which are assumed arbitrarily must be understood to be expressible in the same way by any others which we may assume” (242). Our system of signification could be instantiated in another way and still achieve the same results, but some systems make the truth clearer and easier to see.

If signs are arbitrary does it matter why we choose one set of signs over another? On one hand, Leibniz would answer that ultimately it would not matter what signs you use as long as you use them consistently and do not distort the relationships between

concepts. Leibniz argues that, “Although symbols are arbitrary, their use and connexion has something which is not arbitrary, namely a certain symmetry between symbols and things” (Parkinson 1966, xxv). Arbitrary symbols have the potential to tell us new things about the world. Even though the symbols are arbitrary, this does not mean that the relationships are arbitrary.

On the other hand, Leibniz does think that some signs are better suited for certain types of calculations. He draws a contrast between Arabic and Roman numerals. He says that with both sets of numerals one can perform calculations but that the Arabic numerals are much better suited for calculation than the Roman numerals (Martin 1964, 58). Thus, even though it might not matter which signs are chosen, in theory and in practice, the choice of signs may, and often does, make a difference. The difference lies in how useful our symbolic system is. Certain arbitrary parsigraphic symbols make more evident to our senses the relationships that are inherent to complex concepts than other parsigraphic systems. For example, the five logical operators commonly used in sentential logic are better suited at seeing relationships than the Sheffer Stroke.

The arbitrary nature of the signs allows the Logical Calculus to be pursued with mathematical rigor. For Leibniz knowing that parsigraphic signs are arbitrary is important in understanding how the Universal Characteristic and Logical Calculus could achieve mathematical rigor: “I think that no one who understands these matters doubts that the part of logic which deals with the moods and figures of the syllogism can be reduced to geometrical rigour” (1670, 105). Even though someone might not know what the signs mean, the rigorous nature of the calculations still produces truth. The revealed truth is in the real, meaningful relationships that are seen.

1.3 Logical Calculus

The Universal Characteristic is a great starting point but it does not accomplish all that Leibniz desires this scientific instrument to do. In order to make the Universal Characteristic useful there must be a way to manipulate the symbols and deductively arrive at new theorems. This formulation must also be subject to mathematical rigor in order for Leibniz to be satisfied. This process of symbol manipulation presents new and interesting truths about the world that are not self-evident at first glance. The universal characteristic provides the alphabet and the logical calculus is the rules of syntax. The rules of syntax is not only for the manipulation of arbitrary symbols but also for the understanding of the relationship between the symbols and the world.

Once we have the Universal Characteristic then it will only be a matter of following the rules of the grammar in order to arrive at new deductive truths (Parkinson xvii). The grammatical rules would be able to derive, through deduction, all of the truths that have been produced in the sciences (Russell 1937, 169). It would do more than that though. It would be capable of predicting the results of and directing future research. The question is how the grammar for this language would work. The correct grammar is needed in order to guarantee truth.

Leibniz thought that natural language was not capable of the rigor needed because of the ease with which one can slip into equivocations. The Universal Characteristic would provide a grammar that would avoid equivocation and be equivalent to calculation. Leibniz (1696) said that, "It is not always in our power to find the truth when not enough data are at hand, but we can always guard against error if we have time to think about a

matter and to discover everything possible from the data” (470). The Logical Calculus will provide us with a quick way in determining the truth of an argument. It might not always be in our power to find the truth, but given a statement there would be a simple procedure to see if the argument is correct or not. And if the argument is flawed there will be an easy procedure to see where the argument fails.

Leibniz believed that once the Universal Characteristic was created, one would be able to construct numbers that are consistent with every simple concept (1679, 225). The numbers would provide the logician with the ability to calculate the concepts in order to demonstrate all of the rules of logic. Leibniz (1679) argued that, “A kind of alphabet of human thoughts can be worked out and that everything can be discovered and judged by a comparison of the letters of this alphabet and an analysis of the words made from them” (222). If one calculates according to the rules of the Logical Calculus than one will be able to recognize formal truths. By analyzing form alone, one avoids ambiguity and equivocation.

Leibniz argued that all philosophical arguments needed to be approached using the method of a geometrician. He said (1686b), “I am convinced that if some exact and thoughtful mind were to take the pains to clarify and assimilate their thoughts after the manner of the analytic method of geometricians, he would find a great treasure of very important and strictly demonstrative truths” (309). Here, Leibniz is arguing that if scholasticism was studied in the same manner that geometry is studied then one could find many useful ideas in scholastic philosophy.

An example of the method that Leibniz desires in philosophical discourse is found in Leibniz’s mathematical work. Leibniz (1675) says:

Hence I go on to say that not only can a straight line or a circle, but any curve you please, chosen at random, be taken, so long as the method for drawing tangents to the assumed curve is known; for thus, by the help of it, the equations for the tangents to the given curve can be found. The employment of this method will yield elegant geometrical results that are remarkable for the manner in which long calculation is either avoided or shortened (Child 112).

By arbitrarily choosing a curve, the equation that you are looking for can be found. The geometric method that Leibniz desires is to arbitrarily choose signs and then derive results that produces knowledge. The method of arbitrarily choosing a curve at random is not the only method for solving the equation, but notice that it makes the calculation shorter. The shorter proof is preferred because it is easier to avoid mistake, and if one does make a mistake, it is easier to find where the mistake was made.

The Logical Calculus was designed not only to recognize formal truths or geometric equation but to demonstrate all truths. Leibniz (1679) says that, “From such ideas or definitions, then, there can be demonstrated all truths” (231). This appears to be an overly optimistic goal. Leibniz is claiming that all truths can be demonstrated using a deductive method. For Leibniz, the truths produced by this system are not only probable but are as necessary as the calculations of mathematics. If this is what Leibniz means then the Universal Characteristic is a utopian dream.

The Logical Calculus is based on the idea that all truths are derivable from deductive proofs. But is this the case? Leibniz would argue that all propositions are provable with the deductive method. Several times (1666) he states that, “Every true proposition can be proved” (77). From the simple concepts or axioms, every other truth can be determined by simply calculating correctly. It appears that this system would be

perfect and help further advancement in not only science but in morals, metaphysics, religion etc.

Although Leibniz appears to be dogmatic in his assertion that every truth is provable, he is also modest about certain propositions. In the paper *Necessary and Contingent Truths* (1686c), Leibniz argues that there are two types of truths. There are necessary truths and contingent truths. The necessary truths are those propositions whose opposite leads to a contradiction (Kolak 153). Leibniz argues that in necessary truths the predicate is contained in the subject. These are commonly called analytic truths.

Contingent truths are similar in that the predicate is contained in the subject but one can never demonstrate that this is the case because one can analyze the terms indefinitely. He (1686c) states that, “In such cases it is only God, who comprehends the infinite at once, who can see how the one is in the other, and can understand *a priori* the perfect reason for contingency; in creatures this is supplied *a posteriori*, by experience” (Kolak 153). For God, contingent truths are demonstrable through a deductive argument, but for humans it is not possible because of the limits of our reason. We would have to have an infinite number of steps in our syllogism to prove deductively a contingent truth.

Leibniz argues this same idea in his logical papers. Leibniz makes it clear that when he says that ‘truths are provable’ he is referring to necessary truths, truths which are necessary in all possible worlds, and not for contingent truths. He (1666) says,

Finally, warning must be given that the whole of this art of complications is directed to theorems, or, to propositions which are eternal truths, i.e. which exist, not by the will of God, but by their own nature. But as for all singular propositions which might be called *historical* (e.g. ‘Augustus was emperor of Rome’) or as for *observations* (I.e. propositions such as ‘All European adults have a knowledge of God’ - propositions which are universal, but whose truth has its basis in existence, not in essence, and which are true as if by chance, i.e. by the will of God)- of

these propositions there is no demonstration, but only induction; except that sometimes an observation can be demonstrated through an observation by the mediation of a theorem. (Parkinson 1966, 5-6).

Leibniz makes it clear that there are certain truths which are not provable within this logical system. He does leave open the possibility that there might be a contingent truth that is provable within the system but contingent truths are not necessarily provable within the Logical Calculus.

Leibniz realizes that there are certain propositions that the Logical Calculus will never be able to solve because of the difficulty that one has in finding tautological propositions. Someone could search for an infinite amount of time and still never come to a conclusion for certain contingent propositions. Leibniz does not lament the fact that there are certain undecidable truths. Instead he argues (1686), “But we can no more give the full reason for contingent things than we can constantly follow asymptotes and run through infinite progressions of numbers” (Parkinson 77,78). He does not appear to be disturbed by this limit of the Logical Calculus but is confident that this limit is only a limit of human knowledge. The totality of the real world, in all its infinite complexity, may elude mortals no matter how well-chosen are symbol systems are. However, better systems empower us to see more of the real world than we could without them.

1.4 Leibniz's Legacy

We have seen that Leibniz desired to have a language that reduced all propositions to their simplest terms. This language Leibniz called the Universal Characteristic. He thought that one would be able to turn the simple terms into numbers and perform calculations on the numbers using the grammar of the language. Using this process, Leibniz thought that all true propositions could be proved.

Leibniz's dream for a logical system that was powerful enough to do all these things seems very different from what logic can actually do. However, Leibniz's dream inspired recent mathematicians/logicians to create some robust and powerful logical systems. He has also inspired some great advancement in the understanding of the foundations of mathematics. It is my contention that Leibniz's vision for logic inspired the formation of Boole's Symbolic Logic, Frege's *Begriffsschrift*, and Gödel numbering (which led to the incompleteness theorems).

Bertrand Russell argues that Boole attempted to symbolize and create a grammar which would explain the 'Laws of Thought' (Russell 1967, 170). Russell also argues that Boole's Symbolic Logic was very similar to the vision which Leibniz had for logic. Jean Van Heijenoort also points out that Boole's system accomplishes only the Logical Calculus aspect of Leibniz's dream (Van Heijenoort 1997, 233). Boole made a great start in fulfilling the dream but was not able to provide the Universal Characteristic which was needed in order to fully accomplish what Leibniz desired.

It was Frege who would try to accomplish the Universal Characteristic aspect of Leibniz's vision. In the *Begriffsschrift* (1879), Frege identifies what he is trying to accomplish as a *lingua characterica* and specifically identifies this with Leibniz (Van

Heijenoort 1997, 233). The *lingua characterica* of the *Begriffsschrift* could be identified with the Universal Characteristic of Leibniz. Hintikka says that, “His *Begriffsschrift* was to be primarily a *characteristica universalis* in Leibniz’s senses” (Hintikka 1997, ix). Frege, by expanding the work of Boole, should be seen as strengthening the logical system. After Frege, there is not only a logical grammar to be used in understanding the form of arguments but one is able to create simple terms as part of the Universal Characteristic.

Although Leibniz would have loved the work of Frege he would not have been fully satisfied because the symbols were not yet numerals. One could not use numerals as signs in order to show the certainty that the proofs could offer, the certainty which mathematics does offer. It was not until Kurt Gödel invented the method of Gödel numbering that Leibniz dream was fully realized.

Kurt Gödel was highly influenced by the work of Leibniz. Roman Murawski (2002) asserts that, “Gödel claimed that it was just Leibniz who had mostly influenced his own scientific thinking and activity” (429). Murawski also quotes Hao Wang as saying that all of Gödel’s major works are developments of Leibniz’s ideas (429). Gödel, desiring to complete the dream that Leibniz had for logic, saw that the arguments of logical works could be turned into theorems of mathematics. Gödel took the arguments of *Principia Mathematica* and turned them into numerical calculations. This is called Gödel numbering and aligns perfectly with the vision that Leibniz had for the Logical Calculus. It was Gödel numbering which provided the basis for the incompleteness theorems. Gödel numbering is a prime example of how a powerful system of

signification can show us new things about the world by showing relationship inherent to arithmetic never before seen.

Overall, Leibniz vision for a Universal Characteristic and a Logical Calculus appears at first glance to be a utopian dream but more recent developments in the philosophy of logic and mathematics has shown that Leibniz vision is not one that should be rejected. Symbolic logic and Gödel numbering have been a helpful enterprise in science and mathematics. Leibniz helped provide the vision for further developments in not only philosophy and mathematics, but his vision also provided the foundation for further scientific discoveries. Leibniz should not be dismissed as an overly optimistic dreamer but should be seen as someone who recognizes the limits to human knowledge but also understands the power of mathematical reasoning.

1.5 Leibniz's Continued Relevance

The Universal Characteristic and the Logical Calculus are not only important for abstract mathematics but are also important in our understanding of the world. The ability to reduce all of our language to the simplest terms is a utopian dream that is not achievable, but because part of Leibniz's vision for logic is utopian does not mean we reject the entire project. Leibniz's project enables us to see how understanding the relationships between propositions or objects strengthen our understanding of the world. Leibniz left behind an invaluable foundation that provides us with a means to see that relations are both mathematically representable and metaphysically real.

Leibniz thought that all relations were reducible to arithmetical equations. If you take any proposition, then using the rules of arithmetic the truth of the proposition is

derivable by means of arithmetical steps. Notice that the truth is derivable from the proposition, not just the validity of the argument. I contend that instead certain logical systems can determine the necessary relationships, not the truth, of propositions by means of arithmetic. It is commonly thought that the objects or concepts are the important aspects of our thinking. Instead, I suggest that the concepts in and of themselves are not important. The necessary relationships that exist between concepts or objects are the important aspect of our thought.

There are logical systems that are capable of mathematical representation. Leibniz suggested the idea that logic needs to submit to the same rigor as mathematics and attempted to provide a way in which to complete this idea. Gödel showed that all axiom systems could be turned into arithmetical functions. This idea of Leibniz, which Gödel understood, enables us to see how relations can, and do submit to the same rigor as mathematics.

Leibniz's idea of a Logical Calculus is important because it points out the isomorphism between the way the world is and the way we think about the world. He correctly assessed that the world is bound within certain structures of relations that can be symbolized by using arbitrary signs. Leibniz's theory of relations is metaphysically real. Consider the logical law of identity. There is an isomorphic relationship between $a=a$ and the proposition 'A human is a human.' The letter 'a' is not identical to the word 'human' but both propositions express the same type of relationship that is necessary in the world.

In conclusion, Leibniz's logical ideas are useful in our understanding of the world. Leibniz was a catalyst for the formation of Boolean Logic and of Predicate Logic.

His theory of a Logical Calculus and a Universal Characteristic not only stirred great thinkers and mathematicians in the 20th C. but also provides us with a foundation of understanding the relationships inherent in the world. Understanding these relationships can and does provide useful examination in all of our scientific research and is a scientific tool, as useful for the mind as the telescope is for the eye.

Chapter 2

Foundations of Mathematics

2.1 Introduction

In the early 20th Century, many mathematicians (Peano, Hilbert) and philosophers (Russell, Whitehead) were seeking a way to provide a foundation for arithmetic that would be powerful enough to make it theoretically possible that every mathematical proposition could be solved. They were seeking algorithms and axioms of reasons (as opposed to articles of faith) that would provide a foundational starting point for the entire branch of mathematics. In the early 20th Century, people worked vigorously to provide this foundation. The rigorous search for a foundation led to some interesting, and unexpected results. In examining the history of the search for a foundation of mathematics, I am interested in exploring how new innovative symbolic languages are capable of producing results that are not capable of being seen in any other way. I believe that the search for a foundation of mathematics shows us that there is more to mathematics than mere tautologies.

During this time a system of formal logic was developed that had the qualities that Leibniz desired to see in a logical system. Frege produced his *Begriffsschrift* (1879) that

revolutionized the way that logic was practiced. Previously, Aristotelian syllogisms studied categories and the relationships between categories, but were not powerful enough to take into consideration isolated propositions. Boolean logic took a step in the right direction by providing algebraic notation and symbols but could only express entire propositions with an isolated symbol. With Frege, formal logic was now capable of accomplishing things that at one point it could not even dream of accomplishing, such as explicitly representing the complexity of the kinds of relationships between sentences, predicates, relationships between relationships and properties of relationships. The systems being invented and ideas of this time radically changed the way we think of mathematics and logic.

This new representational power not only fueled the dreams of secure foundations but also led to unexpected discoveries. Philosophers no longer solve the problems of metaphysics in simplistic terms e.g. objects, their properties and relations. Instead, relations define the objects and properties become 1-place relations. For example, the successor relationship defines objects by their relationship to other objects.

Because of this new approach to logic and mathematics, there were some unexpected results that were discovered in the early 20th Century. Two of the surprising results were Cantor's continuum hypothesis, and Gödel's incompleteness theorem. These two discoveries are paradoxical in nature and help shed light on what recursive exploration of formal systems produces. In this chapter, I discuss these two discoveries that helped shape the mathematical world. I am interested in the way in which these episodes illustrate the powerful utility of particular symbolic innovations. For while foundational dreams may seem naïve, in retrospect, the full significance of the formal

discoveries is rarely appreciated for the light they shine on perennial questions in metaphysics and epistemology, while the failure of foundational dreams are well-rehearsed, the epistemological success and philosophical significance of these formal discoveries remain under appreciated.

2.2 Cantor's Diagonal Method

Georg Cantor's most famous work was on the concept of infinity. Mathematicians and philosophers have argued over the nature of infinity since the beginning of time (if there is a beginning...). Boyer (1985) says that, "Ever since the days of Zeno men had been talking about infinity, in theology as well as mathematics, but no one before 1872 had been able to tell exactly what he was talking about" (611). Cantor helped shape our modern understanding of the nature of infinity. Cantor's big idea is that not all infinite sets have the same measure. Some infinite sets are larger than others. Cantor used an innovative method of proof to show his results.

Cantor revolutionized mathematics with the idea that infinity comes in different sizes. He argued that the set of positive integers is much smaller than the set of all rational fractions (Boyer 612). He showed this idea by what is called the diagonalization method. Cantor first assumes that the real numbers between 0 and 1 are countable and then sets them up in a chart.

Mary Tiles (1989) explains the diagonal argument as follows:

	r_1	r_2	r_3	r_4	...
1	0	1	1	0	...
2	0	0	1	1	...
3	1	1	1	0	...
4	1	0	1	1	...
.
.

This diagram shows that the real numbers cannot be fully enumerated. There will always be a number missing from the diagram. Now suppose that all real numbers between 0 and 1 appear on this chart. Now, you could add another row by taking the diagonal list of numbers, change the value for each, and create a new number. Concerning the new number, Tiles says, “This number differs from all those listed because, for each n , it differs in the n th decimal place from r_n ” (110).

The results of this proof show that there is not ‘The infinite’ or a ‘set of all sets’ but that there are many different sizes of infinite sets. In fact, there are an infinite number of different size infinite sets. Cantor’s diagonal method produced interesting results because of the way in which his innovative representational system of proof demonstrates the complexity of real relationships in the world. In the proof he is using the list of reals in order to produce a new list. The diagonal line of numbers is also being used to produce a new number. In this sense the diagonal method refers to itself in order to create a new series. The relative size difference between the reals and the

rationals is a fact about mathematics people could not see until Cantor invented the symbolic means necessary to make the relationship explicit or evident to our senses.

2.3 Gödel's Incompleteness Theorem

Kurt Gödel's paper, *On Formally Undecidable Propositions of Principia Mathematics and Related Systems I* shocked the world of mathematics in 1931. His results were not expected and showed what some took to represent a definitive dead end for the foundation of mathematics. Many people are familiar with what the incompleteness theorem is but are not quite sure what it proves, or how he achieved his results. In this section, I will describe the method of symbolization known as Gödel numbering which was the key in achieving the results of the proof. Gödel numbering made never before seen relationships in metamathematics visible by utilizing a new symbolic method. In this method, existing symbols were used in a new way to refer to strings of symbols.

While some may exaggerate the claims made by Gödel's incompleteness theorem, I will explain what the incompleteness theorem proves about arithmetic as a formal system, and the limits of Gödel's results.

2.3.1 Gödel Numbering

If only true theorems are provable, then a system is consistent. If all true theorems are provable then a system is complete. In order to show the relationship between consistency and completeness in arithmetic, Gödel needed a way to represent metamathematical sentences in the language of mathematics. His solution was to assign a specific number to every symbol in the system. In this manner, any statement in the

symbolic logical notation could be turned into a series of numerals. This innovative idea was foundational for achieving a primitive recursive argument in metamathematical theory. A primitive recursive argument is an argument that is paradoxical because it refers to itself. In referring to itself, a primitive recursive argument is caught in an infinite loop that can never be solved. In what follows, we look at Gödel's paper and what he says concerning Gödel numbering and then look at secondary sources on the significance of his numbering system.

In Gödel's famous paper, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems I* (1931), Gödel begins by giving an informal sketch of the proof and argues for his numbering system. He says, "Proofs, from a formal point of view, are nothing but finite sequences of formulas (with certain specifiable properties). Of course, for metamathematical considerations it does not matter what objects are chosen as primitive signs, and we shall assign natural numbers to this use" (147). First, Gödel point out that metamathematical propositions consist of sequences of mathematical formulas. Then he points out that since the signs used are arbitrary we could assign natural numbers as the signs. Nevertheless, if proofs are no more than formulas, then assigning the natural numbers seems like a justifiable move. Gödel recognizes that all signs are ultimately arbitrary but that well-chosen signs have a greater capacity to show us relationships that are inherent to the world in which we have never seen, by avoiding the ambiguity or equivocations within another language.

Gödel assigns one and only one natural number to every primitive sign in the metamathematical proof. In describing how he is going to accomplish this he says, "We map the primitive signs one-to-one onto some natural numbers" (147). He assigns

distinct odd numbers as primitive signs, for example, the tilde is assigned the number 5 and the open parenthesis is assigned the number 11 (157). In doing so, he assures that there is a unique finite sequence of numbers that corresponds with every finite metamathematical formula (157). He says, “Consequently, a formula will be a finite sequence of natural numbers, and a proof array a finite sequence of finite sequences of natural numbers” (147).

Because Gödel numbers are arbitrary, they can be chosen at random and still produce the same results. Nagel and Newman say that, “There are actually many alternative ways of assigning Gödel numbers, and it is immaterial to the main argument which one is adopted” (69). Because of the arbitrary nature of the Gödel numbers, it allows the proof to remain generalized. The uniqueness of every number also assures that every proposition, sign or proof has a specific number that is unique to that proposition, sign or proof. Obviously, certain numerals would make symbol manipulation simpler. Imagine a Gödel numbering system where the tilde was equal to 1, the universal quantifier equal to 11 and the existential quantifier equal to 111 etc. This system would still be possible, but it makes the task of understanding and arriving at derivations within this system humanly impossible.

What is the benefit of changing our arbitrary signs to natural numbers? Recall the definition of completeness. Only true statements are provable in a consistent system, while in a complete system every true statement is provable. The benefit of Gödel numbering is allowing all statements, provable, true, false or otherwise to have a unique Gödel number. Gödel argues that in doing so, “The metamathematical notions (propositions) thus become notions (propositions) about natural numbers or sequences of

them” (147). Then we acquire the ability to discuss propositions that are about the natural numbers by using the natural numbers. This makes explicit the isomorphism inherent between metamathematics and mathematics. Gödel says, “In other words, the procedure described above yields an isomorphic image of the system PM in the domain of arithmetic, and all metamathematical arguments can just as well be carried out in this isomorphic image” (147). The isomorphism allows us to discuss arithmetic by using arithmetic as our language. Or as Smullyan (1992) says, “The purpose of assigning Gödel numbers to expressions is to enable sentences to talk about expressions indirectly by directly talking about their Gödel numbers” (22).

Notice that one could produce Gödel numbers for more than just metamathematical expressions, because of the isomorphic nature between the Gödel numbers and the formal system or language that one is using. What makes this isomorphic relationship distinct is that it provides a dual-isomorphism. The mechanical rules of the formal system are still present but now there is an added dimension of rules of arithmetic applying to the Gödel numbers. The rules of any formal system, because of the power of Gödel numbering, are identical to arithmetical formulas.

Essentially what Gödel did was what Leibniz suggested be done in the development of logic. Leibniz suggested that if all propositions could be represented by numerals then one could use arithmetic to determine whether the proposition was true or not. In “Two Studies in the Logical Calculus”(1679), Leibniz says, “For example, since man is a rational animal, if the number of animal is a , for instance, 2 and the number of rational is r , for instance 3, the number of man, or h , will be 2×3 or 6” (235). The antecedent condition of Leibniz’s proposal is fulfilled by Gödel numbering. Gödel took

Leibniz's simple idea and showed just how powerful it can be. According to Murawski (2002), "Gödel claimed that it was just Leibniz who had mostly influenced his own scientific thinking and activity" (429).

Another benefit of using the Gödel numbering is that it makes a definite decision procedure for all systems possible. In sentential logic, one definite decision procedure is the truth-trees. But in predicate logic, the truth-trees are useful but not always a decision procedure, because you can have infinite trees when the quantifiers overlap in a certain way. For this reason, the truth-tree method of proof does not provide a definitive decision procedure because it can be endlessly recursive.

Gödel numbering allows a definite decision procedure within a finite number of steps in the same way that factoring is a finite procedure. Gödel numbering allows a proof-pair to be set up between the premises and the conclusion. The proof-pair sets up a primitive recursive relationship between the premises and the conclusion. Because the proof-pairs are primitive recursive it follows that there is a decision procedure for determining if the proof-pair is a theorem or not. The proof-pair gives you a starting point in which there are a finite number of steps in testing for theoremhood. The first step of the decision procedure is to find the Gödel number for the theorem that is in question, which will be a unique number. Once the number is presented there is a formula for the system to determine whether the number is a theorem of the system or not (Nagel and Newman 78). In order to find whether the number is a theorem you are reduced to factoring the Gödel number instead of having to construct a potentially infinite truth-tree. Nagel and Newman state this virtue, "Lies at the heart of Gödel's discovery: typographical properties of long chains of symbols can be talked about in an indirect but

perfectly accurate manner by instead talking about the properties of prime factorizations of large integers” (83). In other words, the isomorphism between the typographical system and arithmetic is a perfectly accurate isomorphism that allows us to decide between theorems and non-theorems. The theorems or non-theorems are not only about numerals but can be translated back to metamathematical statements. Since Gödel numbers are capable of expressing isomorphic relationships between the terms in any language, by using Gödel numbers, there exists a finite decision procedure for all expressions. This feature of Gödel’s innovative use of symbols shows us that it is not the terms or symbols that are important, but the necessary relationships between terms.

2.3.2 Gödel’s Proof

Gödel discovered that by using his method one can come up with a string that not only talks about itself, but also says, “This string is not a theorem of PM” (PM is *Principia Mathematica*). This string refers to itself and is self-referential. The string is almost identical to Russell’s paradox, because of the self-referential nature. PM claimed to have no falsities for theorems. Thus, if “This string is not a theorem of PM” is a theorem then there is a contradiction and if it not a theorem then there is a truth that is not provable within the system.

A possible reaction to this paradox could be, “So what?” A formal system invented by Russell and Whitehead fails to be able to solve every problem that it set out to solve. There is more to the problem than just a flaw with the system of Russell and Whitehead. We should not forget that when we transformed the formula into Gödel numbers the proposition became a statement about arithmetic. The string is not only a

paradox of PM but is a paradox within the nature of arithmetic itself, as a kind of formal system.

What exactly does Gödel's proof prove? Gödel stated in the introduction to his famous paper that:

These two systems are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide any mathematical question that can at all be formally expressed in these systems. It will be shown below that this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integers that cannot be decided on the basis of the axioms. This situation is not in any way due to the special nature of the systems that have been set up, but holds for a wide class of formal systems (145).

Here Gödel argues that his proof shows that there are certain inherent problems with axiom systems in general depending on their scope and not just the systems that he is discussing.

At least three things follow from Gödel's proof. First, Gödel's proof does not state that every formal system is necessarily incomplete, but that every formal axiom system that includes arithmetic is incomplete. A.J. Ayer says that Gödel's argument shows, "That no proof of the consistency of any deductive system, which was rich enough for the expression of arithmetic, could be represented within the system... There is, therefore, a sense in which Gödel proved that arithmetic is essentially incomplete" (130). One could apply the method from Gödel's paper to systems or areas of study in which the proof does not follow. People have taken Gödel's proof to say, "Gödel showed that we can't really prove anything in mathematics!" (Franzen <http://www.sm.luth.se/~torkel/eget/godel.html>). Gödel did not say any such thing because his own proof is an example of mathematics proving something. It is important

to keep in mind that Gödel's results apply only to certain systems that are powerful enough to include number theory. What is philosophically important is that this result is an example of a well-chosen symbol system that revealed to us previously hidden truths.

Second, Gödel's proof shows that there are more mathematical truths than are provable from an axiom system. Nagel and Newman argue that Gödel's proof shows, "That an axiomatic approach to number theory cannot fully characterize the nature of number-theoretical truth" (110). In other words, Gödel shows that mathematics is not a purely deductive science in the sense that the logicist movement thought. Instead, mathematics is more like a scientific study in which new things are discoverable and new methods are capable of producing new truths. If mathematics is like science in this way then Gödel's proof might be pointing us toward mathematical realism. Discoverable truth implies that there exists something that is discoverable. Alternatively, nominalism would argue that mathematics is purely a formal game of symbol manipulation. But if mathematics has the capacity to help us discover new things about the world then mathematics must be more than a mere formal game. Nominalists are correct to point out the arbitrary and conventional nature of our symbols. However, although the symbols are arbitrary, it does not follow that the relationships between them are arbitrary.

Third, Gödel's proof shows that creativity and intuition is just as important, if not more important, in scientific discoveries as the power of research methods and instruments. Nagel and Newman say, "As Gödel's own arguments show, no antecedent limits can be placed on the inventiveness of mathematicians in devising new methods of proof" (110). Gödel invented his own method of numbering in order to create generate his recursive proof. Without his innovative methods, Gödel's incompleteness theorem is

not realizable and we are blind to certain relationships between formal axioms in possible systems. It is only because of innovative systems that we could see Gödel's results clearly.

Gödel's proof shows that there are certain formal systems, those that include number theory, that have undecidable theorems. Gödel's proof radically changed the study of mathematics in the early 20th Century. In order to achieve his results, Gödel devised a way for formal systems to talk about themselves. Gödel's incompleteness theorem revolutionized the search for the foundations of arithmetic in the early 20th Century and helped us understand the limits inherent in axiom systems. The limits of an axiom system need not deter us from using the system. Even though there are inherent limitations on some of our systems, this does not mean that no formal systems are capable of being useful. Cantor and Gödel both show us that formal systems can provide insight into the structure of our world.

2.4 Conclusion

In conclusion, Cantor's continuum hypothesis and Gödel's incompleteness theorem highlight the paradoxical nature of recursive systems. These discoveries helped shape the

mathematical and philosophical world in the 20th C. and continue to shape the 21st C. It is my contention that these discoveries share two things in common with one another.

First, these discoveries point out the usefulness of creative ideas in scientific, mathematical, and philosophical developments. In order to see these paradoxes, a creative new method of proof was employed. Gödel used Gödel numbering, and Cantor used the diagonal method. The discoveries were not possible without the creativity of the proof method that was used. The innovation of the proof strategy shows that mathematical languages are not just arbitrary tags, but that the relationships are inherent to our world.

Second, on the one hand, these discoveries at first were either rejected or seen as a reason to despair. On the other hand, these discoveries help us understand that formal logic and mathematics are more like science than was once thought. Mathematics is capable of discovering new content and is not reducible to axiom systems. As Poincaré (1894) asked, “Are we then to admit that the enunciation of all the theorems with which so many volumes are filled are only indirect ways of saying that A is A?” (382). These two discoveries help us realize that mathematics and logic are not void of content but are actually content-rich. For these two reasons, I do not see the paradoxical nature of Cantor and Gödel as a reason to despair over the loss of a perfect system. Instead, the results of their improvements in method contribute to the advance of logic, mathematics and philosophy by opening our eyes to see the world in new ways.

Chapter 3

Moderate Realism in the Philosophy of Mathematics

3.1 Introduction

Philosophers of mathematics may be loosely sorted into two camps: realism and nominalism (anti-realism). I do not see these as two radically distinct groups, but instead, I propose that there is a continuum between the two groups. On one end of the continuum are Platonic realists, such as Mark Steiner and Kurt Gödel. On the other end of the continuum lie anti-realists such as Michael Dummett and David Hilbert. In this chapter, I will show that both extreme ends of the continuum between realists and nominalists represent a misunderstanding of mathematical relationships.

In order to show the failures of extremes in the realism/nominalism debate, I consider the relevance of Leibniz's logic for 21st Century philosophy of mathematics. In order to make explicit the relationship between Leibniz and 21st C. metaphysics, I first discuss Kant, Frege, Russell and the analytic/synthetic distinction as articulated in the early 20th C. It would be anachronistic to call Leibniz an "analytic philosopher in the Fregean tradition" even though one can selectively quote from his *oeuvres* in such a way that this label makes sense. However, my objections to this label are motivated less by fear of anachronism, or how it obscures the 'real' Leibniz, than inherent limitations within the analytic/synthetic distinction itself. This distinction imposes a rigid divide

between form and content that undermines, or renders meaningless, Leibniz's distinction between the *lingua characteristica*, or a particularly well-chosen system of signification, and the *calculus ratioinator*, or process of reasoning by means of a well-chosen symbol system whereby we learn new things about the real world.

3.2 History of the Analytic/Synthetic Distinction

In order to understand the metaphysics of mathematics, one needs to understand the historic development in the thought of philosophers and mathematicians in regards to the nature of mathematical judgments. Historically, mathematical judgments were divided into arithmetical and geometric judgments. The analytic/synthetic distinction, while first formally articulated by Kant, is implicit in other philosophers prior to Kant. For example, it is possible to see the analytic/synthetic distinction in Leibniz's thoughts concerning necessary and contingent truth, where arithmetical judgments are necessary because they deal with abstract objects, while geometry relates to physical objects in the world. In this section, I will not attempt to find forerunners to the analytic/synthetic distinction prior to Kant, but instead focus on the development and rejection of Kant's ideas in the late 19th and early 20th C.

3.2.1 Kant

In the Critique of Pure Reason (1787), Kant argued that all mathematical judgments (arithmetical and geometrical) were synthetic *a priori*. Kant says that, "All mathematical judgments, without exception are synthetic" (B14). He argues this by examining a

simple arithmetical sentence: $7+5=12$. He argues that at first glance it might appear that this is an analytic statement. Kant says:

But if we look more closely we find that the concept of the sum of 7 and 5 contains nothing save the *union* of the two numbers into one, and in this no thought is being taken as to what that single number may be which *combines both*" (B15 Italics Mine)

According to Kant, it is our intuition that that helps us go outside of the concepts 7 and 5 in order to arrive at the number 12 (B15). What makes arithmetical statements synthetic is the way in which our intuition recognizes the relationship between the two numbers. The union of the two numbers into one takes place in addition, according to Kant. Thus, addition is not concerned with the objects, in and of themselves, but is instead concerned with the relationship between the objects.

Kant argues that this argument is easier to understand if you consider larger number calculations. It is obvious that we cannot possibly count off 3,000 things in our head at once to perform a larger calculation, so there must be a symbolic means of deriving a conclusion in larger sums. Lisa Shabel (2006) argues that, "Kant's point is that, even if we have a shortcut (perhaps symbolic) method for performing large number arithmetic calculations, the *relations* among large number concepts must be justified on intuitive and thus synthetic grounds" (104 Italic Mine). Here we see that mathematical relations are considered the synthetic aspect of mathematics.

The *a priori* nature of mathematical judgments for Kant is self-evident. He says, "It had to be noted that mathematical propositions...are always judgments *a priori*, not empirical; because they carry with them necessity, which cannot be derived from experience" (B15). All necessary truths, like mathematical judgments, are *a priori*.

Kant's view is that mathematical relations are both necessary and universal, but dependent on intuition.

Logic, for Kant, is not the same as mathematics. While mathematical judgments are synthetic in nature, logical judgments are analytic. Concerning logical judgments, Kant argues that

Logic teaches us nothing whatsoever regarding the content of knowledge, but lays down only the formal conditions of agreement with the understanding; and since these conditions can tell us nothing at all as to the objects concerned, any attempt to use this logic as an instrument that professes to extend and enlarge our knowledge can end in nothing but mere talk" (B86/A62).

Notice that for Kant, logic teaches us nothing because it can tell us nothing about the objects but is only concerned with abstract relationships. In the Kantian framework, the objects themselves take priority over the relationships between objects and are our source of synthetic insights.

The problem with Kant's distinction between analytic and synthetic judgments is that Kant considers our logical reasoning to concern objects primarily, while our mathematical reasoning concerns relationships of objects (B86/A62). For Kant logic cannot teach us anything because it cannot tell us anything new about objects. This conclusion does not necessarily follow. Instead, logic does tell us how objects relate with each other because logical truths are isomorphic to mathematical truths. Objects do not have to be our sole source of knowledge in the world. In this way, logical reasoning, by means of an innovative symbol system, can reveal new relationships in the world that we could never have seen before. Once we see logic and mathematics as both concerned primarily with relationships between objects and/or relations then both disciplines can be seen as synthetic in nature.

3.2.2 Frege

Gottlob Frege agreed and disagreed with Kant on the synthetic nature of mathematics. Frege saw geometrical judgments as synthetic but arithmetical judgments as analytic (Tennant 1997). In the *Grundlagen der Arithmetik* (1884), Frege argues that numbers have a specific meaning. Since the concept of number is meaningful, then mathematics, at least in part, can tell us new things about the world.

Frege argues that numbers have meanings but that they are not separate objects or a property of any outward things (87). Frege is worried that this idea might undermine the content of mathematical statements. Instead, he says, “The unimaginability of the content of a word is no reason to deny it any meaning or to exclude it from usage” (88). Just because we cannot imagine what a number might look like, or precisely what we mean when we say the number 2, it does not follow that the word is meaningless. Frege uses an analogical argument at this point. He argues that in the same way that having a mental picture of objects does not mean that our mental pictures are somewhere. We may have an idea of what a number is, but this does not mean that numbers actually exist in some Platonic heaven (89).

Frege argues later that mathematics is both analytic and synthetic. On the one hand, geometrical judgments are synthetic because they can reveal true things about the world. On the other hand, arithmetical judgments are analytic because they can be proved by pure deduction. He says that mathematical inferences expand our knowledge about the world and are thus synthetic, but that they are proved deductively and thus

analytic. Frege's logicism indicates that it is not only arithmetic that is analytic, but because arithmetic is a branch of logic, logic is analytic.

Mary Tiles (1989) argues that, "It was Frege's view that in arithmetic there need be no appeal to any sort of intuition, for arithmetic reduces ultimately to logic in the strong sense that (a) arithmetical objects (numbers) are logical objects and (b) all true statements about these objects can be proved from definitions by appeal only to logical laws" (138). Frege's logicism rests on the assumption that logic and the appeal to logical laws is purely analytic. Arithmetic can be proved deductively, like logical laws, because as we have seen with Gödel numbering, logical statements are isomorphic to arithmetical statements. But just because mathematics and logic can be proved deductively does not mean that their judgments are analytic. The relationships between the logical objects could be real relationships that exist. They do not have to exist in a Platonic universe somewhere, just like mathematical objects do not have to exist in a Platonic universe. Instead, these relationships exist independent of what material or immaterial objects may exist and have meaning.

3.2.3 Russell and the Vienna Circle

In this section, I am lumping Russell in with the Vienna Circle. I know that the argument could be made that Russell's logicism is different than the logicism of Carnap, but those distinctions are not important for the scope of this chapter. The Vienna Circle did not think that any mathematical or logical judgments were synthetic. Every mathematical statement was a tautology and void of any content. Carnap (1931) argues that since all arguments of logic are tautologies it follows that, "all the sentences of mathematics are

tautologies” (142). He says concerning tautologies that, “They say nothing; they have, so-to-speak, zero-content” (143). This is a shift from Frege, because according to the Vienna Circle, not even geometrical judgments classify as synthetic judgments.

Hans Hahn (1933) sees a difficulty that is associated with this view. This difficulty is, “How is it (empiricism) to account for the real validity of logical and mathematical statements?” (149). Hahn argues that logic and mathematics are both composed of no more than tautologies. Therefore, Hahn is in total agreement with Carnap in this respect. This presents a problem for Hahn. Hahn wants to know what exactly the purpose of logic or mathematics is if it is purely analytic. It is Hahn’s belief that tautological statements are important because we are not omniscient (157). If we were omniscient then we would know all tautological statements. Analytic judgments are thus useful because of the limited nature of humans.

Hahn’s argument is weak because the same thing could be said of empirical knowledge. If we were omniscient then we would not need empirical science. We would know the results of the experiment without the need of going through the process. This does not mean that scientific knowledge is without content. The position of the Vienna Circle assumes that tautologies are content-free because they do not know how a tautology could contain content.

Poincaré (1902) argues against the Vienna Circle view that mathematics is not wholly tautological. He asks, “Are we then to admit that the enunciation of all the theorems with which so many volumes are filled are only indirect ways of saying that A is A?” (382). I think that this is the right question to ask. I think Poincaré’s point is adequate even if all of mathematical statements are tautologies, because different

tautologies express different relationships. The statement $A=A$ expresses a different relationship than $A \vee \sim A$, even though it expresses the same truth value. In the same way, even if mathematics is tautological, it is not the case that every theorem expresses the same relationship. If different theorems express different relationships then the tautologies are not void of content, but content-rich.

3.2.4 Quine

In the “Two Dogmas of Empiricism,” Quine (1964) argues that the notion of analyticity is circular. According to Quine, analytic statements are those statements that share cognitive synonymy. One of the problems with analyticity is that analyticity assumes a necessary connection between two objects. Quine does not think that the adverb ‘necessarily’ can account for analyticity. He says, “Does the adverb really make sense? To suppose that it does is to suppose that we have already made satisfactory sense of ‘analytic’” (30). In the essay, Quine shows that all of our notions of analyticity end up relying on synonymy and that our notion of synonymy relies on the notion of analyticity. In this way, the distinction between analytic and synthetic judgments is shown to be a false dichotomy.

Quine is right in pointing out that there is a false dichotomy between analytic/synthetic judgments. He says that the distinction is a “metaphysical article of faith” (37). Instead of holding to this article of faith, Quine proposes that all of our theories are revisable and are in need of reevaluation. Even our logical connections can be revised (42). I think that Quine is correct in rejecting the analytic/synthetic distinction

and that all our theories are revisable. Innovative symbolic systems help us revise our theories by showing us new, interesting relationships in possible worlds.

3.2.5 **The Problem with the Analytic/Synthetic Distinction**

The analytic/synthetic distinction imposes a rigid divide between form and content. Kant saw logic as pure form, and thus analytic; whereas, mathematical judgments, being full of content, are synthetic. The distinction between form and content renders meaningless Leibniz's distinction between the universal characteristic and the calculus ratiocinator. In this section, I show why a rigid divide between form and content misses the important aspect of Leibniz's project. I argue that form and content are not as divided as once thought.

An aspect of Leibniz's project is that there is an isomorphism from a language into mathematics. Gödel numbering shows that this indeed is the case. One can take a language, and translate the components into numerals and manipulate the symbols through the rules of arithmetic. In this sense, the content of a proposition is not important. Instead, in this isomorphic relationship, the form is important because the form shows the inherent relationships that exist between arbitrary objects.

The universal characteristic shows us that the original content of the proposition is no longer the primary content of interest. Instead what was the form becomes the focus of concern, because the terms being used in the argument are arbitrary and may be changed as long as it is done consistently. I do not think that the distinction between form and content is a real distinction. The form/content distinction breaks down because the form contributes to the content of any proposition. In this sense, the form is not

isolated from the content, and the content cannot be isolated from the form. The distinction between analytic/synthetic judgments is meaningless according to this view. All statements, including tautologies, are content-rich because form contributes to content.

The fact that all statements are content-rich allows us to see how a well-chosen symbol system can teach us new things. In the same way that the innovative system of Gödel numbering was able to show that arithmetic is necessarily inconsistent or incomplete, a new system could be invented that opened our eyes to see new relationships in the world. Leibniz's project is not a utopian dream of seeing every theorem proved, but instead shows us that the world of ideas, including the world of mathematics, is filled with content. The content produced by mathematics is dependent on different axiom systems. Different axiom systems produce different content. Mathematics studies all the different relationships in possible worlds and reveals necessary relationships for the instantiation of the world in which we live.

3.3 Nominalism and Realism

Historically, there are two streams in the study of mathematics. On the one side, Platonists argue for the existence of universal, mathematical objects in some platonic universe. On the other extreme, nominalists think mathematical objects are non-existent as mathematics is a complex linguistic game. This section addresses these two streams in light of Leibniz's distinction between the universal characteristic and calculus ratiocinator. Is mathematics about anything? Are the relationships that we have been discussing actually existent, or are they just part of a formal game of symbol

manipulation? Answering these questions allows us to see how mathematical theorems are applicable to other sciences.

3.3.1 Nominalism (anti-realism)

Nominalist philosophers tend to invoke the arbitrary and conventional nature of symbol systems when they argue that mathematics is a purely formal game of symbol manipulation. Two of the branches of nominalism are the intuitionists (Brouwer) and the formalists (Hilbert). Both of these branches see mathematical statements as invented by humans, and thus not real.

Quine, in his discussion of formalism, claims that,

The formalist keeps classical mathematics as a play of insignificant notations. This play of notations can still be of utility...But utility need not imply significance...Nor need the marked success of mathematicians in spinning out theorems, and in finding objective bases for agreement with one another's results, imply significance. For an adequate base for agreement among mathematicians can be found simply in the rules which govern the manipulation of the notations (15).

The formalists see mathematics as a 'play of insignificant notations.' The formalist correctly recognizes the arbitrary nature of mathematical symbols. As we have seen, Leibniz argues that all of our signs are arbitrary and could have been different. Our mathematical notation is arbitrary, but this does not mean that the notation itself is insignificant, especially when it is put to use.

In chapter 1, I argued that mathematical notations are arbitrary. Nevertheless, certain symbols have more utility than other symbols. For example, Arabic numerals are more useful than Roman numerals for doing complex mathematics. In this sense, mathematical notation is arbitrary, yet significant. It does matter what mathematical

notation we use. Some forms of notation allow make relationships evident by linking what were mere thought to be distinct domains.

Quine states that the formalist argues that the agreement between mathematicians is found in the rules of the game that they all play. Mathematicians set up a system with certain axioms, they first agree on the axioms and then manipulate symbols according to these rules. This procedure alone does not explain how mathematics is useful for physical theories. The answer is that mathematicians do not arbitrarily choose axioms. The arbitrary nature of mathematical symbols does not mean that mathematics as a discipline is an arbitrary game. If mathematics is a game of symbol manipulation, then mathematics could not consistently provide an adequate base for scientific discoveries. But mathematics does consistently provide insight for scientific discoveries, so mathematics is not a game of symbol manipulation. What the symbol-sequences of axioms signify makes a difference. Axiom choice, like a choice of signs themselves, is never totally arbitrary.

The intuitionists take a more moderate position than the formalists. The intuitionists see mathematical objects as mental constructs, but mathematics is not purely a formal game. Michael Dummett (1993) says,

The fundamental idea is that a grasp of the meaning of a mathematical statement consists, not in a knowledge of what has to be the case independently of our means of knowing whether it is so, for the statement to be true, but in an ability to recognize, for any mathematical construction, whether or not it constitutes a proof of the statement; an assertion of such a statement is to be construed, not as a claim that it is true, but as a claim that a proof of it exists or can be constructed (70).

The intuitionists hold that to assert that a statement is true is to assert that its proof can be constructed. For the intuitionist, the claim that a mathematical statement is true or not does not make sense outside our awareness of it. Mathematical proofs divide true statements from false ones, but some statements are neither true nor false if there is not yet a proof one way or the other.

The intuitionists, unlike the formalists, see mathematics as having to do with some aspect of truth outside of mental constructions but the problem is that this leads to metaphysical problems. Heyting (1956) says:

In fact all mathematicians and even intuitionists are convinced that in some sense mathematics bear upon eternal truth, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics (57).

For the intuitionist, there is something true about mathematics, but we cannot say what that is. The intuitionists are a step closer to realism, in the sense that mathematics is about something real, but they claim that we have no access to comprehend that reality outside of our mental constructions. The problem with intuitionism is that different mathematicians have different mental constructions, yet mathematicians often agree with one another. This fact points to mathematical truth independent of mental construction.

Overall, the nominalists are correct in pointing out that mathematical notation is largely arbitrary. However, it does not follow from this fact that mathematics is just a formal game. Instead, mathematics is useful and does provide us with insights into the way in which the world functions. Mathematical proofs provide us with insights into the world by allowing us to see relationships in the world that we could not see without an adequate symbolic language.

3.3.2 Realism

Realism in mathematics is typically identified with a radical form of Platonism. This is why Mark Steiner (1975) says, “The fear of Platonism is indeed the primary obstacle in the way of the acceptance of mathematics as a true science” (108). One aspect of Platonism that is universal, is that mathematical Platonists view mathematics as a science that provides its own foundation, as opposed to a science that has its foundation in another branch of inquiry (like logic). In this section, I examine alternative forms of realism in the philosophy of mathematics in order to articulate a moderate version of realism that does not depend on a platonic universe of forms. I examine the ontological Platonism of mathematical objects in the work of Kurt Gödel and Mark Steiner. I then look at Shapiro’s ante rem structuralism. I then present modal relationalism, my moderate view of mathematical realism.

Kurt Gödel (1944), in “Russell’s Mathematical Logic,” argues that the assumption of mathematical objects is just as legitimate as assuming physical bodies (220). He makes an argument by analogy. In the same way that physical laws require physical bodies, mathematical laws require mathematical “bodies” or abstract objects. Gödel quotes Russell as saying, “Logic is concerned with the real world, just as truly as zoology, though with its more abstract and general features” (213). Gödel agrees with the early Russell and sees the method of mathematical knowledge as intuition. Steiner (1975) says that Gödel thought that, “Sets exist, he says, because we intuit them (just as physical objects exist, because we see them) and the existence of such objects renders intelligible questions which are today undecidable” (122). It is mathematical intuition that provides the evidence for mathematical objects.

Steiner calls his realism a more modest version. He says:

There is a more modest way of regarding mathematical intuition. This is to argue...that the existence of mathematical objects is guaranteed by their indispensability to science, by our inability to say what we want to say about the world without quantifying over them (122).

Steiner, like Gödel, believes that the existence of mathematical objects is a science, just like zoology. He argues that arithmetic is its own autonomous science without a foundation in any other branch of inquiry (87). Mathematics is an autonomous science because it studies its own objects.

The mathematical objects that Steiner refers to are numbers. For Steiner numbers exist and are as real as the physical world. He says, “The view that mathematics is a separate science does imply that the natural numbers, the subject matter of the science, are objects in the same sense that molecules are objects” (87). Steiner believes that since mathematics studies the natural numbers, the existence of numbers is just as certain as the existence of physical objects.

There are two things wrong with this claim. First, according to this view, all numbers (real, rational, irrational, imaginary) must constitute arithmetical objects and not just the natural numbers. If numbers are the subject of the science of arithmetic then it follows that every number, not just the naturals, should be existents. Steiner’s argument for the intuition of mathematical objects rested on the fact that the numbers are indispensable for scientific theories. But the natural numbers are not the only numbers that are indispensable for scientific theories.

The second criticism of Steiner’s view is that it claims that the subject of mathematics is the number. I would argue that this is not the case. Applied mathematics never studies numbers in and of themselves. Instead, numbers are always studied in relationship to

other numbers. In this sense, the subject matter of mathematics is not numbers, but the relationships that exist between numbers. I am not denying that numbers might exist, they could exist, but the subject of mathematics is not the numbers themselves, but the relationships between numbers.

The realism of both Gödel and Steiner is headed in the right direction. I think they make the right arguments by analogy. The problem is that they restrict the subject of mathematics to numbers. Steiner is correct by arguing that the indispensability of mathematics is an argument for mathematical content but the content of mathematics is not solely numbers. The numbers are just arbitrary tags while the relationships are not arbitrary. It is the relationships that exist between numbers that makes mathematical judgments indispensable for science.

Stewart Shapiro (1989) adapts a different approach to mathematical realism. Instead of approaching mathematics as a traditional Platonist, Shapiro is a structuralist. Shapiro defines a 'structure' as, "the abstract form of a system, which focuses on the interrelations among the objects" (28). He contrasts traditional Platonism with structuralism by pointing out that the Platonists believe the subject matter of mathematics is abstract objects. Structuralism states that the subject matter of mathematics is patterns or structures (27). On this view, "Arithmetic...is not understood as the study of a particular collection of objects...but rather as the study of the natural number structure" (27). Shapiro is a mathematical realist, but not in the same sense as Gödel and Steiner.

Shapiro believes that if structuralism is adopted that it will make a difference in our mathematical practice, because there is a strong connection between ontology and

mathematical practice (22). He believes that structuralism will provide a foundation for accepting both classical and non-classical logics. He says:

Classical logic is the proper vehicle for studying structure as such, and, in particular, the relations between structures. However, just as the mathematician can (and should) countenance individual systems with alternate logics, the structuralist can countenance individual structures that have alternate logics (32).

The problem with this is that structuralism relies on another structure for its foundation, namely the structure of a particular logic.

Shapiro attempts to clarify this problem. He says, “There is thus a distinction between the logic *internal* to a structure-the logic needed to develop insights about its objects and relations- and the logic used to study how it relates to other structures” (32).

There are two problems with this answer. First, structuralism, in this view, would not provide the subject matter of mathematics. Instead, the logical system that one adopts provides the subject matter for mathematics and then the structures are built up from there.

Secondly, Shapiro makes a distinction between structures and relations. Shapiro indicates that structures relate to other structures. This is because in structuralism, arithmetic is a study of the natural number structure. Shapiro says, “A natural number is not construed as an individual object; on the contrary, a natural number is a place in the natural number structure” (27). It is here that Shapiro makes the same mistake that Gödel and Steiner make, by putting priority on mathematical objects, whether a number or a structure. At first it seems that structuralism is similar to the relational model of mathematics that Leibniz provided a foundation for, but upon further inspection structuralism takes a path similar to traditional Platonism.

For Shapiro, the natural numbers are not the subject of mathematics, but the structure of natural numbers. The difference being that the number 2 does not exist, but that the number 2 occupies a specific place inside of the structure of the natural numbers. Mathematical structures then are no more than place-holders for abstract objects. According to structuralism, addition is the relationship between two different places in the natural number structure. Notice that the subject of mathematics is no longer the structure qua structure, but the relationship of places within a structure. In other words, mathematics studies how places within a relational structure interact with one another.

The failings of the versions of mathematical realism that we have seen does not mean that we must be a nominalist. Instead, I suggest a moderate form of realism called modal relationalism. Modal relationalism, is constructed from the findings of the *lingua characteristic*a and the *calculus ratiocinator* of Leibniz. In examining those two thoughts in Leibniz, we saw that mathematics is about something, but it is not about numbers. It is about the relationships that necessarily exist in the world, and how innovative systems of signification reveal new interesting relationships that we could not see before. In what follows, I weave insights of Gödel, Steiner and Shapiro to construct modal relationalism.

Leibniz claimed that all of our signs are arbitrary tags and thus it does not matter if our signs are numbers or places within the structure of the natural numbers, because the relations will stay the same, no matter what the object. The nominalists are correct in invoking the arbitrary nature of the signs but the realists are correct in assuming that mathematics is more than just a convention. The *lingua characteristic*a and *calculus ratiocinator* of Leibniz show us that mathematical relationships may be real, universal

and necessary-while the symbolic or representational means by which we recognize these relationships are arbitrary or conventional, yet heuristically powerful.

Modal relationalism states that mathematics is the study, not of numbers or structures (even though these might be important aspects of mathematics), but of the relationships between objects in all possible worlds. These relationships do actually exist, but not in some abstract platonic sense. We experience instantiations of these relationships in our world all the time. We see two apples, and then someone takes one and there is only one left. This is a real relationship that is represented by the arbitrary mathematical statement $2-1=1$. In this way, modal relationalism satisfies the intuition of Gödel and Steiner that mathematics is about something and is its own science.

Modal relationalism also explains why mathematics is indispensable for science. The same relations that exist in mathematical theorems are true for scientific theories. Our scientific theories can utilize the relationships that have become evident in mathematical practice.

This version of realism also appeases Shapiro's desire to leave open the possibility of non-classical logics. Modal relationalism studies mathematical relations in all possible worlds, which means that the relationships exhibited by non-classical logical systems are relations in that possible world (which might be our world). Any logical system produces new and interesting relationships that could exist in a possible world. My intuition is that if this model of realism is developed further, one could yet articulate universal relationships that hold true in all mathematical theories, not despite Cantor and Gödel's findings, but because of them.

In conclusion, the distinction between analytic and synthetic judgments is a false dilemma. Every statement, even tautologies, expresses a relationship between possible things. Mathematical language is arbitrary but this does not mean that mathematical theorems are mere conventions. Instead mathematical practice opens our eyes to see new relationships in the world that truly exist, including the relationships that might obtain between possible worlds governed by different axioms. Symbolic language can make evident relationships in our world in the same way that the microscope makes it possible for our eyes to see cellular life.

References

- Ayer, A. J.** 1984. "Philosophy in the Twentieth Century". New York: Vintage Books.
- Benacerraf, Paul and Hilary Putnam.** 1964. "Philosophy of Mathematics: Selected Readings." Englewood Cliffs, N.J.: Prentice-Hall Inc.
- Boyer, Carl B.** 1985. "A History of Mathematics". Princeton: Princeton University Press.
- Carnap, Rudolf.** 1931 "The Old and the New Logic." *Erkenntnis* Vol. 1, 1931. Rpt. in Logical Positivism. Ed. A.J. Ayer. New York: The Free Press, 1959. 133-146.
- Child, J.M.** 1920. "The Early Mathematical Manuscripts of Leibniz." Mineola, New York: Dover Publications.
- Couturat, Louis.** 1903. "*Opuscles et Fragments Inedit De Leibniz*" Paris.
- David, Martin.** 2000. "Engines of Logic: Mathematicians and the Origin of the Computer". New York: W.W. Norton & Company.
- Dummett, Michael.** 1993. "The Seas of Language". Oxford: Clarendon.
- Eco, Umberto.** 1995. "The Search for the Perfect Language". Oxford: Blackwell.
- Franzen, Torkel.** www.sm.luth.se. Lulea University of Technology. 08 May 2009
- Frege, Gottlob.** 1884. "The Concept of Number." Grundlagen der Arithmetik. Breslau. Rpt. In Benacerraf and Putnam 85-112.
- Gödel, Kurt.** 1986. "Kurt Gödel: Collected Works". Ed. Solomon Feferman. Vol. I. New York: Oxford UP.
- Gödel, Kurt.** 1944. "Russell's Mathematical Logic." The Philosophy of Bertrand Russell. *The Library of Living Philosophers*. New York: The Tudor Publishing Company. Rpt. In Benacerraf and Putnam 211-232.

- Hahn, Hans.** 1933. "Logic, Mathematics, and Knowledge of Nature." *Einheitswissenschaft*. Rpt. in Logical Positivism. Ed. A.J. Ayer. New York: The Free Press, 1959. 147-161.
- Heytink, Arend.** 1956. "Disputation." *Intuitionism, An Introduction*. Rpt. in Benacerraf and Putnam 55-65.
- Hintikka, Jaako.** 1997. "Lingua Universalis vs. Calculus Ratiocinator: An Ultimate Presupposition of Twentieth-Century Philosophy". Boston: Kluwer Academic.
- Kant, Immanuel.** 1965. "Critique of Pure Reason". Trans. Norman Kemp Smith. New York: St.Martin's.
- Kolak, Daniel and Garrett Thomson.** 2006. "The Longman Standard History of Modern Philosophy." New York: Pearson/Longman.
- Leibniz, Gottfried W.** 1966. "Logical Papers". Ed. G.H. R. Parkinson. Oxford: Clarendon Press.
- 1969. "Philosophical Papers and Letters". Ed. Leroy E. Loemker. Second ed. Dordrecht-Holland: D. Reidel Company.
- Martin, Gottfried.** 1964. "Leibniz: Logic and Metaphysics". Frome and London: Manchester UP.
- Murawski, Roman.** 2002. "Leibniz's and Kant's Philosophical Ideas and the Development of Hilbert's Programme." Logique and Analyse 179 (2002): 421-37.
- Nagel, Ernest, and James R. Newman.** 2001. "Gödel's Proof". Revised ed. New York: New York UP.
- Peckhaus, Volker.** 2004. "Calculus ratiocinator versus characteristica universalis? The two traditions in logic, revisited." History and Philosophy of Logic 25 (2004): 3-14.
- Poincaré, Henri.** 1964. "On the Nature of Mathematical Reasoning." Science and Hypothesis. New York: Dover Publications. Rpt. In Benacerraf and Putnam 382-389.
- Quine, Williard Van Orman.** 1964. "From a Logical Point of View". Cambridge, Massachusetts: Harvard UP.

- Russell, Bertrand.** 1967. "A Critical Exposition of the Philosophy of Leibniz". London: George Allen & Unwin Ltd.
- Rutherford, Donald.** 1995. "Philosophy and Language in Leibniz." The Cambridge Companion to Leibniz. Ed. Nicholas Jolley. Cambridge: Cambridge UP. 224-69.
- Shabel, Lisa.** 2006. "Kant's Philosophy of Mathematics." *The Cambridge Companion to Kant and Modern Philosophy*. Ed. Paul Guyer. Cambridge: Cambridge UP. 94-128.
- Shapiro, Stewart.** 1989. "Logic, Ontology, Mathematical Practice." *Synthese* 79 (1989): 13-50.
- Smullyan, Raymond M.** 1992. "Gödel's Incompleteness Theorems". New York: Oxford UP.
- Steiner, Mark.** 1975. "Mathematical Knowledge". London: Cornell UP.
- Tennant, Neil.** 1997. "The Taming of the True". Oxford: Oxford UP.
- Tiles, Mary.** 1989. "The Philosophy of Set Theory An Historical Introduction to Cantor's Paradise". Minneapolis: Dover Publications.
- Van Heijenoort, Jean,** ed.1967. From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931". Cambridge, Massachusetts: Harvard UP.
- , 1997. "Logic as Calculus and Logic as Language." Lingua Universalis vs. Calculus Ratiocinator: An Ultimate Presupposition of Twentieth-Century Philosophy. Ed. Jaakko Hintikka. Boston: Kluwer Academic. 233-39.