An initial study to determine a friction-factor model for ground vegetation

Peter Martin Kenney
The University of Toledo

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A Dissertation Entitled

An Initial Study to Determine a Friction-Factor Model for Ground Vegetation

by

Peter Martin Kenney

Submitted in partial fulfillment of the requirements for

the degree of Doctor of Philosophy in Engineering

__________________________________________
Advisor: Dr. Theo G. Keith, Jr.

______________________________
Graduate School

The University of Toledo

December 2009
The University of Toledo

College of Engineering

I HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER
MY SUPERVISION BY Peter Martin Kenney
ENTITLED An Initial Study to Determine a
Friction-Factor Model for Ground Vegetation
BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN ENGINEERING

______________________________________________________________
Dissertation Advisor: Dr. Theo G. Keith, Jr.

Recommendation concurred by

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Dean, College of Engineering
An Abstract of
An Initial Study to Determine a
Friction-Factor Model for Ground Vegetation

By
Peter Martin Kenney

Submitted as partial fulfillment of the requirements for
The Doctor of Philosophy in Engineering
The University of Toledo
December 2009

Fluid drag data for flows through ground vegetation is needed by those studying the atmospheric boundary layer near the earth in wind turbine design, by those performing drainage calculations for fields awash from a great river, by CFD modelers of the lowest levels of a forest fire, and by agriculturalists investigating the wind overturning of crops. All of these researchers lament that there is little plant-drag data available. This work presents an initial study into a plant friction-factor model for a forest-fire simulation CFD program; but hopefully will be useful to the other professionals mentioned.

The model develops a friction factor as a function of Reynolds number for various percentage cover of plants. Artificial plant arrangements were mounted in a clear acrylic frame, backlit from underneath, and, using a mirror, are photographed from sixty feet away. From the photographic data, fractal dimension and an average gap diameter can be determined as a function of percentage coverage. Then the arrangement is placed in a wind tunnel and drag forces on the system measured for upstream velocities ranging up to thirty-eight miles an hour. Most arrangements had axes of greatest and least flow resistance. Averaging the friction-factor values computed for these extremes a friction-factor model is developed for three different artificial plants. These individual models are, in turn, combined into a single model using the fractal dimension data for the plants. Vorticity and flow through the plant canopy causing plant oscillation are not considered.
For my parents

Lura Jean Koons Kenney
Robert Sherman Kenney
Acknowledgements

Many people helped make this project possible. Their assistance is gratefully acknowledged; but, as always any oversights or errors are this person’s alone.

First and foremost, my advisor, Dr. Theo G. Keith, has patiently mentored this work and made available funds to support it. Thank you for your wise counsel and friendship.

Dr. Rodman R. Linn provided the original suggestion for this model.

Frustrating times during in developing the experimental apparatus were shortened thanks to the knowledge and experience of Dr. Terry Ng.

This project would not have been possible without the skills of Randy Reihing, John Jaegly, and Tim Grivanos in the MIME Department’s shop.

Early on this project, it was thought to use cut-and-dried natural vegetation in testing. The debris from such would have left the insides of the wind tunnel less than tidy and the collected cuttings were never used. Nonetheless, recognition and thanks go to Lois and William Brodbeck (City View Farm, Ottawa Lake, MI) who kindly allowed collection of vegetation samples from their farmstead and to John Jaeger (Director of Natural Resources) and Denise Gehring (Director of Environmental Programs) of the Metropolitan Park District of the Toledo Area who issued a collection permit and helped in choosing and identifying species.

New Way Air Bearings made available to the project a Boxway Air Slide at a generous academic discount.
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Chapter 1—Prologue

In 1738, Dutch-Swiss mathematician Daniel Bernoulli (1700-1782) published the principle named after him in his *Hydrodynamica*. This principle states that the fluid frictional head loss (pressure loss) between positions 1 and 2 of Figure 1 in a closed conduit is expressed by

\[ h_l = \left( \frac{V_1^2}{2g} + \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{V_2^2}{2g} + \frac{p_2}{\rho g} + z_2 \right) \]  

(1)

where \( h_l \) is the fluid friction or energy head loss between positions subscripted 1 and 2, \( V \) is the mean velocity, \( g \) is the acceleration of gravity, \( p \) is the fluid static pressure, \( \rho \) is the fluid density, and \( z \) is the elevation of the pipe. [1, 2]

Equation 1 cannot be used to predict head loss in piping design. Instead a system must be constructed and all terms in the equation measured. Thus, engineering design requires a relationship that predicts \( h_l \) as a function of the fluid, the velocity, the pipe diameter, and the type of pipe material. [1]

**Figure 1**—Terms in (Bernoulli’s) Equation 1 [after 1]

---

1 This section relies heavily on Reference 1. It should give the reader a feeling of what is to be presented here in a slightly different arena.
In 1845, a native of Saxony (eastern Germany) named Julius Weisbach (1806-1871) proposed a relation for this fluid head loss which, in the form used today, is

$$h_f = \frac{fL V^2}{D} \frac{\Delta p}{\rho g} \quad \text{or} \quad f = 2 \frac{\Delta p D}{\rho L V^2} \quad (2)$$

where $\Delta p = p_1 - p_2$, $L$ is the pipe length, $D$ is the pipe diameter, and $f$ is a friction factor. The friction factor is a complex function of the pipe relative roughness, $\varepsilon/D$, pipe diameter, fluid kinematic viscosity, $\nu$, and velocity of the flow. Weisbach deduced the influence of roughness, diameter, and velocity on $f$, but the professional community apparently ignored his conclusions. As discussed below, the friction factor was perfected by Henry Darcy (1803-1858) in 1857 and Equation 2 is referred to as the Darcy-Weisbach equation. [1]

Weisbach’s relation was based on eleven of his own measurements and 51 from published data by Claude Couplet (1642-1722), Charles Bossut (1730-1799), Pierre Du Buat (1734-1809), Gaspard Riche de Prony (1755-1839), and Johann Eytelwein (1764-1848). [1]

Weisbach’s work covered most engineering mechanics and set the standard for all later engineering texts—except in France the then center for hydraulic research. The French may have believed that it offered no improvement over Prony’s work. However, Prony’s work was not dimensionally independent; whereas, Weisbach’s friction factor was a dimensionless number. At the time (before calculators), however, Prony’s required fewer mathematical operations. [1]

Around 1770 Antoine Chézy (1718-1798) had published on the fundamental concepts for uniform flow in open channels. From his work, a dimensionally inhomogeneous form of Equation 2 can be established. Unfortunately, his work was lost until 1800 when his former student, Prony, published an account describing it. [1]
By the 1830s the difference between low (laminar) and high (turbulent) velocity flows was becoming apparent. Independently and nearly simultaneously, Gotthilf Hagen (1797-1884) in 1839 and Jean Poiseuille (1799-1869) in 1841 defined low velocity flow in small tubes. In 1857, Darcy also noted the similarity of his low velocity pipe experiments with Poiseuille’s work. [1]

In 1883, Osborne Reynolds (1842-1912) described the transition from laminar to turbulent flow and showed that it could be characterized by the parameter,

\[ \text{Re} = \frac{VD}{v} \]  \hspace{1cm} (3)

where \(\text{Re}\) is now referred to as the Reynolds number. In pipes, flow is usually laminar for \(\text{Re} < 2000\) while turbulent flow generally occurs for \(\text{Re} > 4000\). An awkward region between those two limits is called the critical zone. [1]

Once the mechanics and range on laminar flow were well established, it was a simple matter to equate Prony’s equation and Equation 3 to provide an expression for the Darcy \(f\) in the laminar range as. This is the straight line to the left in Figure 2 labeled Laminar Flow. [1]

\[ f = \frac{64}{\text{Re}} \]  \hspace{1cm} (4)

In 1857, Henry Darcy (1803-1858) published a new form of the Prony equation based on experiments with various types and diameters of pipes. Contrary to existing theory, he
showed conclusively that the pipe friction factor was a function of both the pipe roughness and pipe diameter. Therefore, it is traditional to call \( f \) the Darcy friction factor. [1]

In 1877, J.T. Fanning (1837-1911) was apparently the first to effectively combine Weisbach’s equation with Darcy’s better estimates of the friction factor. He merely published tables of \( f \) values taken from numerous publications, with Darcy being the largest source. It should be noted that Fanning used the hydraulic radius, \( R \), instead of \( D \) in the friction equation. Thus the “Darcy \( f \)” is four times larger than the “Fanning \( f \)” [1].

During the early 20th century, Ludwig Prandtl (1875-1953) and his students Theodor von Kármán (1881-1963), Paul Richard Heinrich Blasius (1883-1970), and Johann Nikuradse (1894-1979) attempted to provide an analytical prediction of the friction factor using Prandtl’s new boundary layer theory. Apparently, Blasius (1913) was the first person to apply similarity theory to establish that \( f \) is a function of the Reynolds number. Using data from Nikuradse, the entire turbulent flow range is better fit by the relationship,

\[
\frac{1}{\sqrt{f}} = 2 \log(\text{Re}\sqrt{f}) - 0.08. \tag{5}
\]

It is plotted on Figure 2 and labeled *Smooth Pipes*. [1]

Rough pipes offered additional challenges. At high Reynolds number in rough pipes, \( f \) becomes a constant that is only a function of the relative roughness, \( \varepsilon/D \), where \( \varepsilon \) is the root-to-tip height of the interior pipe-roughness asperities. Similar to the smooth pipe formula, von Kármán (1930) developed an equation confirmed by data collected by Nikuradse (1933),

\[
\frac{1}{\sqrt{f}} = 1.14 - 2 \log \left( \frac{\varepsilon}{D} \right). \tag{6}
\]

The horizontal lines on the right of Figure 2 plot Equation 6 for various ratios of \( \varepsilon/D \). [1]
The transition region between laminar and fully turbulent rough pipe flow was defined empirically by detailed measurements carried out by Nikuradse (1933) on pipes that had a uniform roughness. His data showed clear trends that could be explained by the interaction of the pipe roughness with the fluid boundary layer. For commercial pipes Cyril F. Colebrook (1939) showed the transition region could be described by,

\[
\frac{1}{\sqrt{f}} = 1.14 - 2\log \left( \frac{\varepsilon}{D} + \frac{9.35}{\text{Re} \sqrt{f}} \right).
\]  

Equation 7 is also called the Colebrook-White equation and is implicit thus requiring many iterations to solve for the Darcy-Weisbach friction factor. It is plotted in Figure 2 for various ratios of \( \varepsilon/D \) in the region labeled Transition Zone. [1, 2]

Finally, in 1942, Hunter Rouse (1906-1996) integrated these various formulas into a useful structure. He noted that the equations were far too complex for practical use and that a diagram could be produced to incorporate all of the formulas. Rouse’s original contribution in addition to the overall synthesis was defining the boundary between the transition and fully turbulent zones as,

\[
\frac{1}{\sqrt{f}} = \frac{\varepsilon \text{ Re}}{D 200}.
\]

Equation 8 is plotted on Figure 2. [1]

Lewis Moody (1880-1953) was in the audience when Rouse presented his paper in 1943. Moody felt that Rouse’s diagram was “inconvenient” and decided to redraw it in the more conventional form. [1]

It should be noted that Moody’s diagram is more convenient to use when finding \( h_l \) with known flow rate and \( D \). However, Rouse’s diagram allows a direct, noniterative solution for flow rate with known \( h_l \) and \( D \). Thus, each has its advantages. [1]
Rouse in 1942 appears to be the first to call Equation 2 “Darcy-Weisbach”; but, that designation did not become well accepted by American authors until the late 1980s. [1]

Rather ironically and contrarily to the practice with the equation name, the $f$ versus $Re$ diagram is universally credited to Moody, and the contributions of others are seldom acknowledged—a sore point with Rouse. [1]

Efforts have been made to streamline the procedure and eliminate the manual use of graphs. Thus, in 1976, the most notable advance in the application of the Darcy-Weisbach equation has been the publication by Prabhata K. Swamee and Akalank K. Jain of explicit equations for pipe diameter, head loss, and the discharge through a pipe. These equations for fully-flowing pipes are based on the implicit Colebrook-White equation. Their work used Johan Frederik Seffensen’s root-finding method. The Swamee-Jain equation for friction factor is [1, 2]

$$f = \frac{0.25}{\log_{10} \left( \frac{1}{3.7 D} \frac{\varepsilon}{\frac{15.74}{Re^{0.9}}} \right)^2}. \quad (9)$$

The Darcy-Weisbach equation combined with the supplementary Moody Diagram is the accepted method to calculate energy losses resulting from fluid motion in pipes and other closed conduits. With it piping systems may be analyzed and designed for any fluid under most conditions of engineering interest. [1]
Chapter 2—Nomenclature

$A_f$— frontal area of vegetation. For a single plant specie, one would expect a percentage-cover-type function to govern this variable. The function would vary depending upon the horizontal distance into the vegetation being considered. This horizontally-viewed percentage cover should also be directly related to the vertically viewed percentage cover, $PCT$, for a given plant specie. See the development of the variable vegetation density, $VegD$, in Appendix A. Neglecting the constant relating vertical and horizontal percentage cover and also the depth into the vegetation, $L$, frontal area of vegetation is

$$= H \times W \times PCT$$

$A_o$— frontal open area around and between plants and leaves as seen by the flow.

Same assumption as for $A_f$

$$= H \times W \times (1 - PCT)$$

$d_f$— fractal dimension of plant arrangement (or, for a negative image, the gaps in between)

$d_{\Box}$— box-counting dimension used to approximate $d_f$

$D$— characteristic dimension—either a pipe diameter in the Darcy-Weisbach equation or the mean distance of an equivalent circular gap from the lacunarity calculation.
\( D_{Av} \) – an equivalent diameter based on the average number of pixels in a lacunarity calculation

\( E \) – Euclidian (topological) dimension

\( f \) – friction factor used in the Darcy-Weisbach equation

\( f_{eff} \) – effective friction factor used in considering vegetation

\( g \) – acceleration of gravity

\( h_l \) – head loss in energy equation

\( H \) – height of vegetation and of test section

\( = 27 \) inches

\( L \) – length of vegetation and of test section

\( = 33\frac{15}{16} \) inches

\( n(s,r) \) – frequency distribution of the number of boxes of size \( r \) containing \( s \) black pixels

\( N(\varepsilon) \) – number of \( \varepsilon \times \varepsilon \) grid squares to cover a fractal image in the box-counting algorithm

\( N(r) \) – total number of box masses of size \( r \) in the gliding-box algorithm, \( N(r) = (M - r + 1)^2 \).

\( p \) – fluid static pressure

\( PCT \) – plan-view percentage cover of vegetation

\( PPI \) – pixels per inch—used to scale dimensions form photos

\( Q(s,r) \) – probability distribution, of the number of boxes of size \( r \) containing \( s \) black pixels, \( Q(s, r) = n(s, r)/N(r) \).

\( r \) – edge length of box used in gliding-box algorithm

\( R \) – hydraulic radius

\( \text{Re} \) – Reynolds number
\( s \) – mass of black pixels encompassed by the box of size \( r \)

\( s_s \) – standard deviation of number of black pixels in gliding-box algorithm—the variance is \( s_s^2 \)

\( \bar{s} \) – mean of number of black pixels in gliding-box algorithm

\( V \) – fluid velocity

\( W \) – width of vegetation and of test section

\[ = 33\frac{1}{16} \text{ inches} \]

\( z \) – elevation of point in flow where taking measurements

\( Z^{(1)} \) – first statistical moment of pixels in gliding-box algorithm, equal to the mean value, \( \bar{s} \).

\( Z^{(2)} \) – second statistical moment of pixels in gliding-box algorithm, equal to

\[ s_s^2(r) + (\bar{s}(r))^2. \]

\( \Delta p \) – difference in pressure between two points in the flow

\( \Delta p_{ps} \) – Pitot-static tube pressure difference

\( \Lambda \) – lacunarity

\( \varepsilon \) – edge length of a single grid square, see \( N(\varepsilon) \)

– asperity root-to-tip roughness of inside of pipe

\( \mu \) – fluid dynamic viscosity

\( \nu \) – fluid kinematic viscosity equal to \( \mu/\rho \)

\( \rho \) – fluid density
Chapter 3—Introduction

The impetus to this project is as follows: One day this person’s advisor, Dr. Theo Keith, came to a meeting with his, then eleven, graduate students and said that he had just heard from one of his first Ph.D. students, Ms. P.J. Smith. He said that she is retired, lives in southern California, and is a volunteer smoke watcher during the wildfire season. Then Dr. Keith asked rhetorically: I wonder if anyone has ever modeled a forest fire using computational fluid dynamics (CFD)?

This sounded like interesting research and this person spent the next few weeks searching out if such a model existed.

Wildfire computer models such as the USDA Forest Service’s FARSITE do exist that predict wildfire intensity, speed, and direction. Most of these programs curve fit statistical/empirical data to model a fire; but they do not intimately couple the wildfire with local atmospheric motions which affect, and are affected by, a wild fire.

However, the FIRETEC model does use CFD. It is the 1997 Ph.D. work of Rodman Linn at Los Alamos National Lab (LANL) [3, see also 4]. It evolved after fourteen fire fighters died in the 1994 South Canyon fire near Glenwood Springs, CO. This prompted LANL and other agencies to develop better computer models including FIRETEC, RAMS (Regional Atmospheric Modeling System, Colo. State University), and HIGRAD (Hi-Resolution Model for Strong Gradient Applications, large-eddy model by Jon Reisner at LANL). [5]
Realizing that major inroads had been made in wildland\textsuperscript{2,3}-fire CFD modeling at LANL, this person thought to direct his efforts to helping to perfect that work. When asked what topics related to FIRETEC needed polishing, Dr. Linn stated that the grass drag model needed much work. The term grass includes all ground vegetation. This person agreed to look into the subject.

The term grass drag as it pertains to wildfires needs some explaining as the reader is, no doubt, asking: Why would one need to know a grass friction factor for a forest-fire model when the wind driving a fire forward is not hindered by the already burnt grass? The answer is that a fire draws air into itself from all directions —including through the unburnt vegetation downwind of the fire.

The need for drag data for grasses, forbs\textsuperscript{4}, \textit{etc.}—and the lament of the lack thereof—exists also in the studies of flood-water drainage from fields and through plant-filled, drainage channels \cite{8}, of lodging\textsuperscript{5} in crops \cite{9, 10}, of soil erosion wind-breaks, and of inter- and intraplant air flow in wildland fire simulation. This work was performed to address the latter; but hopefully, will be useful in the other arenas.

The objective of this work is to develop a friction factor as a function of Reynolds number model for flow through ground vegetation of varying plant arrangements using a

\begin{footnotesize}
\begin{itemize}
\item \textsuperscript{2} As some of the terminology of this paper may be unfamiliar to an engineer, some definitions have been supplied.
\item \textsuperscript{3} \textbf{wildland}: Rangelands cover a broad category of land characterized by native plant communities that are often associated with grazing. Such lands are managed by ecological as to agronomic methods. One half of all US land \cite{6} and 60\% of the landmass of the world is classified as rangeland \cite{7}. This value does not consider small grains under cultivation. Collectively, non-urbanized, uncultivated lands of forests, grasslands, rangelands are considered wildlands.
\item \textsuperscript{4} \textbf{forb}: a broad-leaved herb other than a grass, esp. one growing in a field, prairie, or meadow.\[<\text{Greek phorβē}, fodder, <pherbein, to graze]\n\item \textsuperscript{5} \textbf{lodging}: the flattening of a plant or crop stand by strong wind, usually involving root lodging (uprooting) or stem breakage. Root lodging occurs early in the season when soils are very wet and the roots cannot support the plant. It is the most common type of lodging. Stem breakage, on the other hand occurs later in the season as the stalk becomes more brittle due to crop maturation. \cite{12, 13}
\end{itemize}
\end{footnotesize}
variable familiar to ecologist. Said differently, a Moody-type diagram and/or a Swamee-Jain-type equation is desired for flow through ground vegetation.

A clarification is needed here. Whenever a flowing fluid is in contact with a solid object, the fluid transfers some of its momentum via *skin-friction drag* to that object. Such is the case in the pipe flow discussed in the prologue. This frictional force varies directly with the fluid’s dynamic viscosity and velocity as well as the relative roughness of the inside of the pipe.

When flow is around an object, another type of drag must also be considered. This is called *form drag* or *pressure drag*. When a viscous flow first encounters an object it wants to travel along the surface of that object. However, because of the increase in static pressure within the flow as it continues to pass over the object, the flow separates—*i.e.*, moves away from the body. This separation forms a wake region behind the object characterized by turbulence and relatively lower pressure. The net difference in pressure forces between the upstream portions of the object and the downstream wake region is the form drag. Streamlining an object causes the flow to remain attached longer and the wake region and pressure drag are reduced.

Moody’s diagram has been used as an example because roughness is involved. In plants, the “roughness” is caused by the shape of the individual specie as well its arrangement if more than one plant is considered—skin friction on leaves and stems plays only a minor role in total drag.

---

*separation:* increasing the fluid pressure is akin to increasing the potential energy of the fluid, leading to a reduced kinetic energy and a deceleration of the fluid. Since the fluid in the inner part of the boundary layer is relatively slower, it is strongly affected by the increasing pressure gradient. For a large enough pressure increase, the fluid may slow to zero velocity or even become reversed. When reversal occurs, the flow is said to be separated from the surface. [2]
Chapter 4—Theory

This chapter introduces the reader to those concepts that are relevant to this experiment and the computation of its results.

4.1—Types of flow

Two different types of flow are considered through vegetation [11]—flow when the vegetation is completely submerged in the fluid medium and flow when the vegetation is only partially submerged as shown in Figure 3. When the vegetation is fully submerged,

![Diagram of types of flow through vegetation](image)

Figure 3—Types of flow through vegetation [after 16]. The upper figure of fully-submerged vegetation shows vortices dipping down into the vegetation causing it to wave back and forth. The lower figure shows partially-submerged vegetation and the flow has no vertical component.

there is flow through its canopy\(^7\) caused by vortices. These vortices cause wheat to wave and aquatic vegetation to oscillate. As this work was developed to determine a vegetal friction factor for a forest-fire CFD program, only the partially submerged condition is considered as the CFD program is interested in flow \textit{through} a computational cell containing and having the same height as the vegetation. A CFD program can generate vortices and thus

\(^7\) canopy: the stratum containing the crowns (tops) of the tallest vegetation present, (living or dead). [14] The layer formed naturally by the leaves and branches of trees and plants. [15]
flow through the vegetation’s canopy; but the existing and future wind tunnels used to
measure plant drag cannot generate vortices because of the limited height and length of the
wind-tunnel test section and because of honeycombing used to straighten flow upstream.
Thus, this study will investigate partially-submerged flow through vegetation.

4.2—Percentage cover

Percentage coverage, $PCT$, is a variable used by ecologists to determine vegetation
density. It can be determined from aerial surveys and will be the base variable for this
work. Consider Figure 4 where individually, Leaves $A$, $B$, and $C$ occupy 31.5, 19.6, and
7.8% (sum = 58.9%) of the reference area (quadrat\(^8\)); but because higher leaves
overshadow lower leaves, they collectively cover only cover 53.5%, which is the value of
percentage cover. It should be obvious that this variable indicates neither the foliage
height nor the vertical distribution of vegetation.

![Figure 4](image_url)

**Figure 4**—Percentage cover. The umbrella plant (*umbella sinistra*)
used here is found in Costa Rica and other parts of Latin America.

Reference 8 recommends the use of percentage cover and leaf-area index (*LAI*, a one-
sided leaf area per volume, see Appendix A) as variables to be used in studying fluid
friction through vegetation. The variables presented next eliminate the need for
considering *LAI*.

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\(^8\) **quadrat**: *n.* a sampling plot, usually 1m\(^2\), used to study plant or animal life.
4.3—Fractal dimension

In an attempt to create a vegetal Moody-type diagram it was imagined that the “relative roughness” might correspond to the fractal dimension, \( d_f \), which is discussed briefly here and developed at length in Appendix B.

Fractal geometry is a way of measuring the roughness of an object or phenomenon. B.B. Mandelbrot says: “Most of nature’s irregular and fragmented patterns can be described by a family of shapes called fractals. A fractal approach is often times both effective and natural allowing the investigator to perceive the hidden order in the seemingly disordered, the plan in the unplanned, or the regular pattern in the irregularity and roughness of nature. Yet, it is not a panacea.” [17] He further states: “… when some real thing is found to be unsmooth, the next mathematical model to try is fractal ... A complicated phenomenon need not be fractal, but finding that a phenomenon is not even fractal is bad news.” [18]

The box-counting dimension, \( d_\Box \), is a convenient way of approximating the fractal dimension. To calculate the box-counting dimension for a fractal, imagine the fractal image lying on an evenly-spaced grid, and count how many grid squares or boxes are required to cover the set. The box-counting dimension is calculated by seeing how this number changes as one makes the grid finer. Suppose that \( N(\varepsilon) \) is the number of boxes of side length \( \varepsilon \) required to cover the set. Then the box-counting dimension for a particular box size is defined as [2]

\[
d_\Box = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}. \tag{10}
\]

After a number of calculations using different box sizes, one can construct a log-log plot the number of grid squares to cover a fractal, \( N \), vs. 1/box size, 1/\( \varepsilon \). This yields a plot such as shown in Figure 5. The slope of the plot is the box-counting (fractal) dimension.
Figure 5—Results for fractal dimension calculation. The slope of the line is the fractal dimension.

Fractal dimension is usually a noninteger and greater than the topological dimension of the object considered. The Mathcad algorithm for computing this can be found in Appendix C.

4.4—Lacunarity

Two objects can have the same fractal dimension and yet appear to have different textures. The difference is in the gaps or intervals in the objects image. Lacunarity refers to the gappiness or hole-iness of the object. An object—whether a fractal or not—is termed lacunar if its gaps or intervals tend to be large and/or if the gap sizes are distributed over a greater range. [17, 19]

A number of algorithms have been suggested to determine the lacunarity statistic. The gliding-box algorithm used in this study was originally described by Allain and Cloitre [19] and analyses the mass distribution of a set. It is a straightforward algorithm for calculating the lacunarity of both deterministic and random fractals. [20]
Lacunarity and its calculation using the gliding-box algorithm will be demonstrated by example. Place a $2 \times 2$ pixel box in the upper left-hand corner a grid of black-and-white pixels forming a fractal image. Let the black pixels be assigned a mass of 1 and the white ones a mass of 0. The $2 \times 2$ box is just large enough to focus on 4 pixels arranged $2 \times 2 (r = 2)$. Count the mass of black pixels encompassed by the box, $s$. Then the box is caused to glide to the right by one pixel and the box mass again determined. When the gliding box reaches the end of a row, it is returned to the left and moved down one pixel and the process is repeated until the entire image has been considered. The total number of box masses of size $r$ determined will be

$$N(r) = (M - r + 1)^2.$$  \hspace{1cm} (11)

A frequency distribution, $n(s, r)$, of the number of boxes of size $r$ containing $s$ black pixels can then be determined. This frequency distribution is then converted into a probability distribution, $Q(s, r)$, by dividing by $N(r)$; i.e.

$$Q(s, r) = n(s, r)/N(r).$$  \hspace{1cm} (12)

Then the first and second moments of the distribution are

$$Z^{(1)} = \sum s Q(s, r) \quad Z^{(2)} = \sum s^2 Q(s, r).$$  \hspace{1cm} (13)

Finally, the lacunarity, $\Lambda(r)$, is defined as

$$\Lambda(r) = Z^{(2)}/(Z^{(1)})^2.$$  \hspace{1cm} (14)

The statistical moments can also be written as

$$Z^{(1)} = \bar{s}(r), \quad Z^{(2)} = \bar{s}^2(r) + (\bar{s}(r))^2,$$  \hspace{1cm} (15)

where, $\bar{s}(r)$ is the mean and $\bar{s}^2(r)$ the variance of the number of sites per box. Thus, lacunarity can also be written in more familiar terms as
The lacunarity statistic is thus a dimensionless representation of the variance. [19, 20]

The computed value of lacunarity varies with box size, $r$, as shown in Figure 6.

![Figure 6—Lacunarity, $\Lambda$, vs. gliding-box size, $r$, calculation for the plant arrangement of Figure 5](image)

Reference 20 states: “As described by Allain and Cloitre (1991), the lacunarity curve for self similar fractals should be a straight line with a slope equal to $d_f - E$, where $d_f$ and $E$ are the fractal and Euclidean (topological) dimensions respectively.” Here $E$ equals 2.

In Figure 6, a line of slope $d_f - E$ has been added to demonstrate this. That the lacunarity curve does not conform to this line for small gliding-box sizes, $r$, is because the image being used is not self-similar and even for digital images of self-similar (mathematical as to natural fractal) constructs the self similarity is lost due to the finite size of pixels (see example at the bottom of third page of Appendix C).

### 4.5—Characteristic dimension

This work desires to produce a friction factor along the lines of that used in the Darcy-Weisbach equation (Equation 2). Thus, a characteristic length dimension, $D$, is

$$\Lambda(r) = \left( \frac{\bar{s}(r)}{\langle \bar{s}(r) \rangle} \right)^2 + 1.$$  \hspace{1cm} (16)
needed. In the Darcy-Weisbach equation, $D$ represents the diameter of a pipe. Further, this characteristic dimension is needed to compute Reynolds number (Equation 3).

Originally, this study envisioned using some dimensional form of lacunarity of the plant-arrangement gaps to represent of this dimension. As part of the lacunarity calculations of Appendix C, the mean value (for a given gliding-box size) of the number of black pixels in the box is computed. Knowing the number of pixels per inch, $PPI$, in the photographic data, one can determine the average diameter for the gaps, $D_{Av}$. This value will be used as the characteristic dimension.

### 4.6—Dimensional analysis

Here the assumption that the percentage cover as determined from a plan-view photograph is also representative of the plant arrangement as viewed horizontally. This assumption should be true for a single plant specie where each plant has the same silhouette.

With this assumption, the vegetation frontal area, \( i.e. \), what the approaching fluid sees, is defined as \( A_f = H \times W \times PCT \); where the height and width of vegetation (and the test section) are \( H \) and \( W \) respectively.

In calculating velocity through the vegetation the relation

$$ V = \frac{1}{1 - PCT} \left[ \left( \frac{2 \Delta P_{Pr}}{\rho} \right)^{1/2} \right] $$

(17)

is used. The term in square brackets is the upstream (without vegetation) fluid velocity as determined by the Pitot-static tube pressure differential, $\Delta P_{Pr}$. The term $1/(1 - PCT)$ begins to add the influence of the vegetation. For a single plant type, it should suffice to describe the velocity for varying percentage cover. However, this relation fails when $PCT = 1$ (100%) for there would still be a finite velocity around and between the plant stems below the leaves.
(see Figure 7). This region below the leaves (for the plants tested here) and the open region above the plants as they bend under load (see Figure 9e) are the main avenues of flow for moderate to heavy vegetation.

Other variables of interest to a dimensional analysis are the average gap diameter, $D_{Av}$, static pressure drop across the test section, $\Delta p$, the length of the test section, $L$, the fluid velocity, $V$, the fluid density, $\rho$, and the fluid dynamic viscosity, $\mu$. A dimensional analysis then leads to the following dimensionless quantities

$$\frac{\Delta p}{L} \frac{D_{Av}}{\rho V^2}, \frac{pV D_{Av}}{\mu}, \text{ and } \frac{A_f}{D_{Av}^2}. \quad (18)$$

Fractal dimension, $d_f$, the lacunarity statistic, $\Lambda$, and percentage cover, $PCT$, are already dimensionless terms.

The first two terms are a Darcy-type friction factor and the Reynolds number (Equations 2 and 3) for a single pipe of length $L$. In the third term, instead of frontal area of vegetation, consider the frontal open area, $A_o$ as this represents the open area the fluid actually flows through. Also include the constant $\pi/4$. Then, the third term becomes

$$4A_o/\pi D_{Av}^2 = 4HW(1 - PCT)/\pi D_{Av}^2$$

This multiplier represents the number of “pipes” of diameter $D_{Av}$ that could be placed in the open spaces around and between plants and leaves. In essence, this multiplier extends the single pipe of length $L$ in the friction factor to one long

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9 Equation 2 for the Darcy friction factor, $f = 2 \frac{\Delta p D}{\rho L V^2}$, $\frac{\Delta p D L}{\frac{1}{2} \rho V^2}$, has the coefficient 2 where no such coefficient is being used here. If the three terms within one of the pair of brackets on the right-hand side of Equation 1 were converted to pressure terms by multiplying by $\rho g$, one would have $\frac{1}{2} \rho V^2 + p + \rho g z$. The first term is the dynamic pressure, the second is the static pressure, and third is the elevation (head) pressure. The convention of adding the two to Equation 1 is so that the change in static pressure, $\Delta p$, is scaled against the dynamic pressure.
pipe of length $L[4HW(1 – PCT)/\pi D_{Av}^2]$. This work will not carry along the constant $4/\pi$; but, will define the effective friction factor as

$$f_{eff} = \frac{\Delta p D_{Av}}{L \rho V^2} \frac{D_{Av}^2}{HW(1 – PCT)}$$

(19)

4.7—Preferred axis of flow

Cultivated crops are planted in a pattern. For example, corn is traditionally planted about every six inches in rows 30 inches apart. [21] Obviously, the resistance to flow through such crops is highly dependent to the direction of the flow relative to the rows. Wildland vegetation is sown by a different Hand such that its arrangement would be called random by an investigator. Yet, even such random plantings will have, at least locally, axes of greatest and least flow resistance.
Chapter 5—Experimental procedure

The experiment can be broken into two major parts: 1) the collecting and processing of photographic data of various arrangements and 2) the measurement of drag and average upstream fluid velocity for these same arrangements.

5.1—Photographic data

Various arrangements of the plants of Figure 7 were investigated. For each arrangement percentage cover, fractal dimension, lacunarity, and the average gap diameter discussed above were determined using Mathcad algorithms (Appendix C) to process digital, back-lit, plan-view, black-and-white photographs of plant arrangements. The procedure for taking back-lighted photographs is demonstrated in Figure 8. The light-green leaves of Plant 1 did not offer strong contrast and the plants were spray painted dark green. A back-lit photo such
as shown in the upper right-hand corner of Figure 5 is not, *per se*, what is desired—it represents the mass of the plant and not the open space around and between plants and leaves that the fluid actually passes through. This work uses the value of percentage cover from such a positive photo image; but, the fractal dimension, lacunarity, and average gap size are determined using the negative image of plant photos.

### 5.2—Force and velocity measurements

Originally, two different types of shear-beam load cells were tried. These were to measure drag force and also to support an acrylic base with plants. Unfortunately, such load cells do not respond well when the vertical load from supporting the acrylic base is present. Then the experiment was placed on rollers which would support the arrangement and allow it to move against a load cell. Alas, such an arrangement had too much friction and repeatable results were not achieved.
Finally, it was decided to use an air bearing to support the acrylic base with plants and allow virtually friction-free movement against a load cell. The air bearing is shown as Item 2 of Figure 9a and is a Boxway Air Slide by New Way Air Bearings. Item 1 of that figure is an aluminum plate to which is attached the air bearing. This plate is later secured to the floor of the wind-tunnel test section.

In order to calibrate the load cell, an AirCore 2 spectra cord was attached to the air bearing and passed over a pulley (Item 3) after which known weights were attached (Item 4). Early choices in a pulley offered too much friction and the one shown was constructed of two New Way half-inch air bushings with their mounting blocks which are used to support a half-inch shaft. Each bushing can support 20 pounds with 90 psi supply air.

The load cell used is shown in the lower right-hand insert (Item 5) is an Omega AirCore 2 spectra cord was attached to the air bearing and passed over a pulley (Item 3) after which known weights were attached (Item 4). Early choices in a pulley offered too much friction and the one shown was constructed of two New Way half-inch air bushings with their mounting blocks which are used to support a half-inch shaft. Each bushing can support 20 pounds with 90 psi supply air.

The load cell used is shown in the lower right-hand insert (Item 5) is an Omega
LCEB-50 Minibeam Load Cell. Its signal passes through an SCXI box (Item 6). The output signal from this is passed to the computer where a LabVIEW program written by Dr. Terry Ng was then used to determine average load cell strains.

The overall arrangement shown in Figure 9b is termed an insert wind tunnel as it is inserted within a larger wind tunnel. The components of this insert wind tunnel are:

Item 1 — The acrylic base for mounted artificial plants which is mounted atop the air bearing.

Item 2 — A tapered section to reshape the nozzle of the larger wind tunnel as this apparatus otherwise introduces a 6-inch step in the flow. Note that the otherwise uniform flow entering the insert wind tunnel is expected to be faster at the lower part of the test section because of this tapered section.

Item 3 — A frame holding 1/4-inch honeycombing which ensures straight flow before Item 4 which encloses the test section. A single Pitot-static tube was extended through the honeycombing at its center to determine average upstream velocity. Because of the very low pressure difference, a sensitive instrument (not shown) is required to measure it. This pressure difference was measured with Dwyer 475-000-FM Mk III hand-held digital manometer. This manometer measures pressure differences up to 1 inch w.c. with an accuracy of 0.001 inch. Earlier a Dwyer Model 1425 hook gage was used to measure differential pressure. Because of the damping properties of the liquid, this device gave a non-fluctuating average reading once equilibrium was reached. However, because of the small diameter of the Pitot tube, equilibrium took roughly 15 minutes to establish.

Item 4 — A larger frame that spans the Plexiglas plant support. This is a remnant from an early objective of trying to measure the change in total pressure across the test section.
This insert wind tunnel was placed into the test section of the University of Toledo’s low-speed wind tunnel (Figures 9c and 9d). This existing wind tunnel has a 3ft × 3ft test section and was utilized for its fan, which can produce velocities in excess of 150mph.

Early in this work in a phone conversation with Dr. Linn, he stated that the maximum velocity he expected to see would be around 10 m/s. In order to make data for this initial work attractive to those investigating flood-water drainage, a report by Utah State University [22] for the Army Corps of Engineers was referred to. This investigation seemed to use a maximum water velocity of 3.6 ft/s (1.1 m/s). Using Reynolds number scaling to convert this to an air velocity yielded 17.1 m/s (38.3 mph). This is the maximum upstream air velocity investigated in this work.

At least four samplings of velocity and force were made for each plant arrangement over a range of wind tunnel speeds. The wind tunnel speed was selected and allowed to become constant. Because of fluctuating readings of the hand-held manometer, the average of the high and low pressure readings over a 2-3 minute period was used to determine an average Pitot-static pressure difference. This assumes that such readings are Gaussian.

As to force measurement, a calibration curve relating load-cell outputted voltage to force was established. This was verified periodically throughout the experiment and is reported in Appendix D. The air-bearing pulley was located at the center line of the wind tunnel at location A of Figure 9d.

After measuring the Pitot-static pressure difference for a particular wind tunnel speed, the LabVIEW program was executed five times successively—see Figure 10. The average output voltage was taken as the average of the greatest and least values. Loadings both by calibration weights and wind-tunnel speed were consistently preformed such that the loads
only increased. No reverse loadings (decreasing the loads back to zero) were considered. Drift in the data acquisition equipment was less than $5.6 \times 10^{-7}$ volt over 20 minutes. This corresponds to a load of $8.28 \times 10^{-3}$ lbf (less than the weight of 4 grams). On average, one loading sequence took less than 15 minutes.
Chapter 6—Results

As with the procedure, this chapter is divided into two parts: 1) The determination of percentage cover, fractal dimension, lacunarity, and an average gap diameter from photographic data; and 2) The determination of a friction-factor vs. Reynolds-number (force and velocity calculations) model for these arrangements.

6.1—Photographic results

Calculations of percentage cover, fractal dimension, lacunarity, and a mean gap diameter for each plant arrangement investigated are to be found in Appendix C. Calculations were made for both positive and negative images. Combined results for positive and negative images are shown in Pages 184 and 185. On those pages, summaries of all computed values are presented. Of particular interest to this study are the results of the negative images; i.e., calculations based on the gaps between plants and leaves where the flow actually passes. Plots of the negative image calculations as a function of the actual percentage cover, PCT, are shown in Figure 11. Again, the value of PCT used here is that computed from positive photographic images.

Plot (a) of Figure 11 is of fractal dimension, $d_f$, vs. percentage cover, $PCT$. The data for each of the three plants investigated (see Figure 7) regressed nicely into the lines reported with the plot. For a given plant species, if one knew just one value of fractal dimension (of the gaps) at a particular percentage cover, a reasonable guess to other values of fractal
dimension vs. percentage cover can be made by drawing a line through the known point and the point \((PCT = 0, d_f = 2)\).

Plot \((b)\) is of lacunarity based on a 50-pixel gliding box vs. percentage cover. The data for Plant 3 (data indicated by an \(\times\)) might be regressed into a separate curve; but, since data points for all three plants seem to follow closely to one another only one overall regressed curve was determined and is reported with the plot at this time.

Further investigation may reveal that separate curves should be used.

Everything just said about Plot \((b)\) also applies to Plot \((c)\) except now lacunarity is computed with a 100-pixel gliding box.

As lacunarity does not offer a characteristic dimension for use in dimensionless analysis, the mean diameter of the gaps based a 50-pixel gliding box are reported in Plot \((d)\). The data has dimensions of inches and appear to have less scatter than those of Plot \((b)\).

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Figure 11—Plots of fractal dimension, lacunarity, and mean gap diameter for negative images vs. percentage cover based on positive images. Lacunarity values are for a gliding-box sizes of 50 and 100 pixels. Mean gap diameters are based on the same lacunarity calculations.
Everything just said about Plot (b) also applies to Plot (c) except now lacunarity is computed with a 100-pixel gliding box.

As lacunarity does not offer a characteristic dimension for use in dimensionless analysis, the mean diameter of the gaps based a 50-pixel gliding box are reported in Plot (d). The data has dimensions of inches and appear to have less scatter than those of Plot (b).

Plot (e) is the same as Plot (d) except a 100-pixel gliding box is now being considered.

All of these plots and reported regressed equations indicate that these values are functions of percentage cover, which is, as stated, a variable familiar to ecologists and easily determined from aerial surveys.

Note that the photographs processed in Appendix C needed to be edited to remove shadows such as shown in Figure 8e. These shadows particularly occurred where the plant leaves rested against the walls of the light box. A brighter back light than the six fluorescent tubes used is needed along with a better diffuser of the light.

Note also that arrangements which were identical except for their orientation produced slightly different values of percentage cover mainly due to foliage resting against the side of the light box as shown in Figure 12. Differences in other calculated values existed for the same reason.

![Arrangement 1J and 1K](image)

**Figure 12**—Identical arrangements differing only in orientation produced different values of percentage cover due to vegetation resting against the sides of the light box.
6.2 — Wind-tunnel friction-factor results

The material in this section is divided into how the data was initially processed and the actual results.

6.2.1 — Initial data reduction

All raw data is to be found Appendix D along with plots for each plant arrangement of effective friction factor vs. Reynolds number. Because the desired wind tunnel speeds were never exactly duplicated with each sampling, it was originally thought to average the values of $f_{eff}$ and Re within a cluster of data points in order to establish an average curve for a particular plant arrangement. For dense foliage arrangements such as 3G, this produced unacceptable values for a 95% certainty interval based upon Student’s $t$-distribution. The circled groups of calculated points in Figure 13 are the clusters being considered. The insert at the upper, right-hand corner of the figure is a plot of the certainty for the friction factor as a percentage of the mean friction factor value. In the figure, along with the four clusters of

![Figure 13](image-url) — Initial plot of friction factor, $f_{eff}$ versus Reynolds number, Re, calculated results for five samplings with four loadings for each sampling of Arrangement 3G.

![Figure 14](image-url) — Friction factor vs Reynolds number curve after adjusting the calculated friction factors to coincide with the average Reynolds numbers.

![Figure 15](image-url) — Improved certainty interval after discarding first data sampling.
calculated values, are shown the error bar (95% certainty intervals) plots for each average Reynolds number and the average friction-factor curve for the 5 samplings as a solid line.

In the figure, the points plotted using a circle (O) represent the calculated points for one sampling. A cubic spline has been fitted to these points and is indicated as a dotted line. Using the average values of Reynolds number and this spline fit, values for the friction factor can be computed for each average Reynolds number. This was done for each sample using the average Reynolds numbers. The resulting plot is shown in Figure 14: notice the reduction in the 95% certainty intervals.

After running the first sample for a particular arrangement, the next was begun when the Pitot-static pressure reading went below 0.003 inches wc (the lowest non zero reading possible on the hand-held manometer). This turns out to correspond to an upstream velocity of 1.1 m/s—so there was usually a slight load when starting the next sampling. To make all the data consistent, the first sampling for each arrangement was usually discarded. Doing such further enhances the confidence interval as shown in Figure 15 but reports slightly reduced values of friction factor. Depending on the arrangement, sometimes discarding the first sampling increased the certainty interval because of reduced number of samplings. Most arrangements produced certainty intervals about half of those shown in Figure 15.

6.2.2—Friction-factor curves and model development

This study tested various arrangements of increasing percentage cover for three different plants. In calculating the results presented here, fractal dimension and average gap diameter were determined from the curves of Figure 11. To a limited extent, the influence of axes of least and greatest flow resistance is investigated. Figure 16 displays the overall
Figure 16—Effective-friction-factor vs. Reynolds-number curves for the three plants investigated. Arrangement miniatures are used to label the curves. When two identical arrangements orientated to achieve the greatest and least values of friction factor have been investigated, the friction factors of the two arrangements have been averaged and the miniatures displayed together. Averaged data are plotted as Os. All other data are plotted using Xs.

results. A single plant such as in Arrangement 1A does not have a preferred axis. Some arrangements such as 1B and 1C begin to look at directionality by rotating the arrangement 45° relative to the flow. In Figure 16, averaged curves of these similar
arrangements are presented. Notice that the curves are virtually linear on the log-log plots. It is assumed that the curves want to be linear and any nonlinearity shown is due to the quality of the data. The nonlinearity for low Reynolds numbers is attributed to the sensitivity of the load cell. The curves may not be truly linear if one considers the Moody diagram of Figure 2; but locally (over small ranges of Reynolds numbers) the curves of Figure 2 are also close to linear.

Using averaged values for identical arrangements orientated at different angles to the flow, Figure 17 looks at how the friction factors of all three families of curves vary with percentage cover for a Reynolds number of $4 \times 10^4$. On each of the plots, note that, for increasing Reynolds number, the points initially plot linearly and then seem to fall below the line drawn for Plants 1 and 2 and, probably, above it for Plant 3. Also notice that Plant 3 has decreasing friction factor with increasing percentage cover.
Figure 17—Variance in effective friction factor vs. percentage cover for a Reynolds number of $4 \times 10^4$. Points indicated by Os indicate averaged values of least and greatest values of friction factor. Notice how the points plot linearly over much of the range.
Figure 18—The same data as presented in Figure 17 except identical plant arrangements at varying orientations to the flow have not be averaged.

To address flow directionality, a far greater body of data needs to be generated than is presented in this work. However, Figure 18 begins to qualitatively address this subject. Here the same basic data as Figure 17 is presented without averaging similar arrangements. In the
figure, ◇s have been used to denote arrangements that should, by inspection of the arrangement miniatures, offer greatest flow resistance and ✗s least resistance. All other arrangements are denoted with a +.

The dashed lines of Figure 18 are qualitative and should not be taken as true bounds of the friction factor as an arrangement is rotated relative to the flow. To demonstrate this, compare the distances between Arrangement 2B and 2C and between Arrangement 2D and 2E in the figure. The dashed lines do seem to demonstrate the falling away of the friction factor with increasing Reynolds number for Plants 1 and 2.

As the plotted log-log curves of Figure 16 are linear, it would seem a simple model might be constructed if one could determine how the slopes and intercepts of these lines varied with percentage cover. Figure 19 plots the slopes of the curves of Figure 16 while Figure 20 plots the intercepts.

In Figure 19, notice that the points for low Reynolds numbers seem to plot as a straight lines passing through the point (1, 1) in the upper, right-hand corner of each plot. This is most noticeable for Plant 1. This is just an observation and has not been given further consideration. As the curves of Figure 16 are not truly linear, the slope and intercept of the curves depend on which points are used to determine the values. Here the second, third, and forth points were used.

As the slopes and most of the intercepts were negative, the negative of these values were plotted in Figures 19 and 20.
Figure 19—Plots of the slopes of the curves of Figure 16
Figure 20—Plots of the intercepts of the curves of Figure 16

In Figure 20, there is no trend in the intercept values for Plants 2 and 3. However, it is interesting to note that the array of points for Plant 1 is virtually the same as the plot of the slopes for the same plant (Figure 19) except the image is inverted.
The material thus far presented may be helpful to future studies; but, the failure to determine even a weak relation defining the intercept of the log-log curves of Figure 16 forces this project to attempt a different model. The curves of Figure 16 are a function of two variables: Reynolds number and percentage cover. At this time, flow orientation cannot be handled. Thus, using only the averaged results of greatest and least friction factor, when such exist, the procedure of Appendix E will be used to form a solution for each plant type of the form

\[ f_{\text{eff}} = c_0 + c_1 \text{Re} + c_2 \text{Re}^2 + c_3 \text{PCT} + c_4 \text{PCT Re} \]
\[ + c_5 \text{PCT Re}^2 + c_6 \text{PCT}^2 + c_7 \text{PCT}^2 \text{Re} + c_8 \text{PCT}^2 \text{Re}^2 \]  

(18)

The results of this model are shown in Figure 21.

For each plant in Figure 21, the curves whose data is denoted by a circle (○) were used to generate the coefficients. For Plant 1, substituting the listed coefficients into Equation 19 gave good results in describing the curve for Arrangement 1B/C; but it also described the curve for Arrangement 1J/K.
Figure 21—Individual models for each plant
When the coefficients are substituted into Equation 19 for each plant type and plotted along with the other information of Figure 17, an interesting pattern is seen that seems to verify the falling away of the data from the straight lines of that figure. This is shown in Figure 22—an attempt to demonstrate this was the rational for Figure 18.

Figure 22—Plots of individual models overlaid on data for $Re = 4 \times 10^4$
In producing the curve for Plant 3 in Figure 21, a dummy point between Arrangement 3A and 3D/E was used. This is the large circle at a percentage cover of 0.40 in Figure 22. Otherwise two of the three averaged points were too close together and generated the curve of Figure 23.

![Figure 23](image)

**Figure 23**—Points used to determine individual model coefficients should be taken from the linear range. Otherwise, a curve as shown here passing through the three points indicated by Os is generated.

In attempting to forecast the friction-factor behavior of other arrangements, the coefficients presented in Figure 21 did not produce agreement. This is probably because of the differing axes of preferred flow for these arrangements. It may be that there are lines similar to the dashed lines of Figure 22 for other orientations. These might be called meridians of orientation. When future data is taken, arrangements might be placed on a turntable and roses (as in a wind rose) of friction factor developed.

This work needs a method of relating the three different models of Figure 21 to one another. The only tool available to differentiate plants is the fractal dimension curves of Figure 11a. Since fractal dimension is a linear function of percentage cover, its value for a percentage cover of 1 (100%) will be used. Functions for each of the 9 coefficients can then
be generated as a function of this fractal dimension value for each plant type. These functions are shown in Figure 24.

Figure 24—Functions defining coefficients of combined model. Plant type is distinguished by the value of fractal dimension for $PCT = 100$.

If it is not already obvious to the reader, this work has been using fractal dimension all along. For example in Figure 22, friction factor is plotted as a function of percentage cover; but, it has been shown in Figure 11a that the fractal dimension of a given plant type is a linear function of the percentage cover.
Chapter 7—Discussion

Although there has already been some discussion in the presentation of the results just above, a few further comments are needed.

Here only the three plants of Figure 7 were investigated. These all have a common silhouette in that they all get wider along the vertical length of the plant. However, some plants such as the goldenrod of Figure 25 have leaves radiating all along the length of its stem; and others plants are fuller nearer the base of the plant. An investigation into plant shapes is needed.

Above, it was assumed that the friction factor for horizontal flow is the same as that for vertical flow through the canopy. This of course is not true. Flow through a dense canopy would be greatly restricted. Further, from inspecting Figure 3, as vegetation waves due to vortices, the resistance to flow rising up through the canopy would less than down through it.

This work assumed that the percentage cover of a plan-view image of the vegetation was indicative of the percentage cover as realized in a horizontal view. This seems true for a particular species. This assumption was used in constructing the effective friction factor where one long pipe $HW(1 - PCT)/D_{Av}^2$ times longer than the test section was assumed. This equivalent pipe length needs to be better studied in relation to percentage cover and also with respect to the average gap diameter, $D_{Av}$.
Average gap diameter precipitated from the lacunarity calculation for a 100-pixel gliding box. This is a box about 2.25 inches on a side. However, in many instances, the actual plant gaps exceeded this value. A better measure of the gaps between plants and leaves is needed.

Of course, both horizontally viewed percentage cover and average gap diameter need further study with respect to arrangement orientation relative to the flow.

Studies should be made with vegetation of mixed species. Also, actual scaling tests should be carried out to verify, say, that the effective friction factor would be half as large if plants were twice as tall.

Finally, the influence of plant deformation needs to be investigated. The plants might be restrained from bending using fine wires stretched across the test section.
Chapter 8—Conclusions

This work has shown that a Moody-type “convenient form” diagram for a ground-vegetation friction factor does exist and can be expressed in equation form given the arrangement’s percentage cover, its flow orientation relative to the plant stand, and a knowledge of the plant type. Here, the fractal dimension has been substituted Moody’s relative roughness either directly when combining models or indirectly as fractal dimension is a linear function of percentage cover. The experiment used plants of similar silhouette; but, plants having a different overall shape would most likely have a different set of equations defining the coefficients of Equation 19.

As stated in the introduction, this model should be of interest to those persons investigating flood-water drainage, wind overturning of crops, soil erosion wind-breaks, and wildfires. This research should also be extendable to larger plants—i.e., trees. There does seem to be more research, data, and funding for trees and shrubs (but no model per se) than for ground vegetation.
Chapter 9—Future work

Dried or artificial plants may be arranged on a fixed base and placed in a wind tunnel in the laboratory and the vegetal drag force can be measured by attaching load cells to the fixed base. But, how does one measure the drag force on foliage outside the laboratory?—load cells cannot be attached to plants or soil.

Experiments to verify that the difference in total pressure across the test section can be equated to the drag force on the vegetation need to be undertaken. In theory, this should be true; but, the small Pitot-static differential pressure and the small drag forces expected will make obtaining accurate data challenging using field-worthy sensors.

If such a method of determining vegetal drag can be perfected, then data from actual native plant communities could be collected. This would lead to a rapid and relatively inexpensive growth of the overall plant-drag database. In contrast, the U.S. Army Engineers contracted Utah State University to use its 510ft long flume with potted plants mounted on load cells over a 75ft test section to determine plant drag. [22] The flume is 8ft wide and 6ft deep. Such data collection is time, labor, and equipment intensive.

The sampling in the field would use portable wind tunnel along the lines of that shown in Figure 26. Such would probably be hoisted from a flat-bed semi trailer with the fan remaining on the trailer and a flexible duct delivering the flow. The various components of the apparatus are explained in the figure. Some type of outer shell that is not structurally connected to the actual wind tunnel would be needed to protect this wind tunnel from wind loading on its outer surface.
Figure 26—In-field wind tunnels. (a) Envisioned portable wind tunnel to measure vegetation drag. (b) Its size would be something fitting on a semi trailer and might look like this one build by Sterling (Univ. of Birmingham (UK) crop lodging tunnel [10]) (c) The tunnel would be hoisted from the trailer with the fan remaining on the trailer as with the shown USDA Agricultural Research Service’s wind-erosion tunnel. [23] (d) Another small tunnel by Kansas State University’s wind erosion tunnel [24] An outer jacket with no load transfer to item ‡ would be needed to prevent ambient wind loadings from being measured by the load cells.
Of course, sampling in the field allows the actual roughness of the ground to be taken into account—soil is not truly flat, plants have senescent early vegetation leaves at their bases as shown in Figure 25, and there is often other debris around the base of plants.

A smaller laboratory version of this wind tunnel would initially be perfected. Using artificial plants, this apparatus in itself would increase the available data. This along with studies of flow through the canopy and a better understanding of the average gap diameter for horizontal flow would enhance the procedure used here before attempting to collect data in the field.

This work and the future wind tunnels just discussed only consider flow through vegetation—the partially-submerged case of Figure 3. In order to consider the fully-submerged case of that figure, laboratory experiments could be performed using a flume where water is the testing fluid. The depth of water would become an additional variable to be considered in the dimensional analysis. Using the flume, the depth of water could be varied from less than or equal to the plant height (partially-submerged case) to much, much greater than the plant height to model the case where the atmosphere is the submerging fluid. Such a variety of fluid depths should satisfy the needs of all possible users of this future model. Because of the size of the vortices envisioned and because laboratory space is often limited, miniature plants could be used. This would also reduce the pumping requirement for the flume. Upstream plant arrangements identical to the arrangement in way of the test section could be used to preshape the velocity profile before actually encountering the test section. The only other variables to consider are the stiffness and damping of the plants.
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Appendix A—Vegetation Density Indices

Assume that vegetation (stems, tillers, leaves, etc.; i.e., the whole plant) is uniformly distributed horizontally and vertically such that if a very long pin were extended into the vegetation, the pin would contact, on average, $m$ vegetal elements in a distance $D$. One asks: What type of vegetation densities can be developed from this model? [1]

Fischenich and Dudley [1] discuss three such density indices. These are: percent cover ($p$), leaf-area index (LAI), and vegetation density, ($V_{eg}d$). The first two can be determined by satellite and airborne sensors to one degree or another and the remote sensing community is working to improve capabilities—particularly in achieving finer resolution, viewing the understory\(^{10}\) of trees, and, eventually, the identification of individual species. If forest fire simulation software is ever to be used tactically in a faster-than-real-time manner, then finely-scaled, remotely sensed fuel/vegetation maps must be downloadable. It is assumed that remote-sensing community will reach their goal and that such density data will eventually be available to fire managers. Besides satellite/aerial sensing, there are also a number hand held units used to determine percent cover and LAI (mainly for trees in a forest setting) from ground level. These vegetation indices are next reviewed along with their underlying probability theory\(^{11}\).

\(^{10}\) understory: younger trees and other plants beneath the forest canopy.

\(^{11}\) It is assumed the reader has some background in probability and this work will summarily review only those topics of interest to it. Otherwise, see any basic text on probability; e.g., Reference [2].
A.1 Basic density indices

Percent cover, \( p \), is the most easily understood. If a \textit{quadrat}\(^\text{12}\) is observed from above, that percent of the downward view that is not soil is the percent cover. In Figure 1, is shown an umbrella plant having three leaves. Individually, Leaves \( A, B, \) and \( C \) occupy 31.5, 19.6, and 7.8\% (sum = 58.9\%) of the quadrat; but because higher leaves overshadow lower leaves, they collectively cover only cover 53.5\%, which is the value of percent cover. It should be obvious that this measure does not indicate the vertical distribution of vegetation and that foliage of other plants might also overlap.

Thus, in the long pin analogy presented above, one is \textit{not} interested in the average number of contacts, \( m \), after full penetration of the flora. Rather, this model considers only if there is at least one contact or none (soil). Also, no direct mention is made of the height of the vegetation, \( h \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The umbrella plant (\textit{umbella sinistra}) is found in Costa Rica and other parts of Latin America.}
\end{figure}

In this work, a value of percent cover is easily determined by taking a back-lighted, plan-view, digital, black-and-white photograph of the vegetation and then reading that image into a Mathcad matrix. The individual matrix elements can then be easily read and averaged between white (255) and black (0) to yield a percent cover. Although such a photo is essentially

\(^{12}\) \textit{quadrat}: \textit{n. ecol.} a sampling plot, usually \( 1m^2 \), used to study and analyze plant or animal life.
black and white, each pixel of the matrix is examined and, if it contains some shade of gray, set either to black or white (see Appendix D). This is the approach of Reference [3].

The second vegetation index is the leaf-area index (LAI)$^{13}$. This is the ratio of the total, one-sided, vegetation leaf area in a quadrat to the area of the quadrat. “Imagine all the leaves lying on the ground of a deciduous forest in the autumn. If there are, on average, three layers of leaves on the ground in a $1km^2$ area, then the LAI would be $3km^2$ of leaf area per $1km^2$. If a pin were randomly inserted through the leaves, then an average of three leaves would be pierced.” [1] In the percent cover model, the pin was advanced until it hits either vegetation or soil. Now if the probe contacts vegetation, it is advanced further and may contact yet other plant parts. Concisely, LAI is

$$\text{LAI} = \frac{\text{Number of Hits with Vegetation}}{\text{Number of Points (Probes)}}$$ (1)

When using satellite or airborne surveys to determine LAI, elaborate algorithms are used to measure the sunlight or laser reflection from photosynthetically active (green) plant parts (leaves and stems of some plants like cattails). But, this study needs a simple laboratory (and

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$^{13}$ **Leaf area index (LAI):** “is a dimensionless variable and was first defined as the total one-sided area of photosynthetic tissue per unit ground surface area (Watson, 1947). For broad-leaved trees with flat leaves, this definition is applicable because both sides of a leaf have the same surface area. However, if foliage elements are not flat, but wrinkled, bent or rolled, the one-sided area is not clearly defined. The same problem exists for coniferous trees, as needles may be cylindrical or semi-cylindrical (Chen and Black, 1992). Some authors therefore proposed a projected leaf area in order to take into account the irregular form of needles and leaves (Smith, 1991; Bolstad and Gower, 1990). However, in this case the choice of projection angle is decisive, and a vertical projection does not necessarily result in the highest values. Myneni et al. (1997) consequently defined LAI as the maximum projected leaf area per unit ground surface area. Within the context of the computation of the total radiation interception area of plant elements, and based on calculations of the mean projection coefficients of several convex and concave objects of different angular distributions, Lang et al. (1991) and Chen and Black (1992) suggested that half the total interception area per unit ground surface area would be a more suitable definition of LAI for non-flat leaves than projected leaf area. Their theoretical reasoning behind abandoning the projection concept was that the latter has neither physical nor biological significance, whereas the total intercepting area has a physical meaning (e.g. radiation interception) and the total area has a biological connotation (e.g. gas exchange). Still other definitions and interpretations of LAI have been proposed. These vary depending on the technique used to measure LAI. So, in current literature and next to Watson’s definition, LAI defined as one half the total leaf area per unit ground surface area is being used (Chen and Black, 1991; Chen et al., 1991; Fassnacht et al., 1994; Stenberg et al., 1994). It is important to note that these different definitions can result in significant differences between calculated LAI values.” [4]
field) measure for grasses; and, the digital camera method used in this work can not distinguish stem from leaf. So using the same general definition as used for LAI, one can also define a *plant-* and *stem-area indices*, PAI and SAI, such that

\[ PAI = SAI + LAI \]  

(2)

PAI\textsuperscript{14,15} is used in this study.

**A.2 Binomial and Poisson distributions**

Before discussing a PAI model, a review of the binomial and Poisson distributions is in order. The *binomial distribution* (Jas. Bernoulli\textsuperscript{16}) is used in cases where there are only two possible outcomes; e.g., the heads or tails of a coin toss. For percent cover, a single *Bernoulli trial* is when a probe is advanced until it contacts either vegetation or bare soil. In general, the two outcomes are termed *successes* (heads or vegetation present) and *failures* (tails or soil encountered). The *binomial probability of success*, \( p \), is the average number of successes in a given number of trials, \( N \). The actual value of percent cover is the *probability of success in contacting vegetation*, *i.e.*,

\[ \frac{\text{Average number of first contacts with vegetation in } N \text{ trials}}{\text{(Total number of trials, } N)} = p = \text{Percent Cover} \]  

(3)

\textsuperscript{14} The original works on *apparent foliage density* only considered ground vegetation—*i.e.*, the complete plant. Later works addressed trees and they seem to be the focus of the greater body of research both in terms of vegetation density and also drag forces [4].

\textsuperscript{15} Alternative terms for PAI are *Vegetation Area Index*, VAI (Fassnacht et al., 1994), *Plant Area Index*, PAI (Neumann et al., 1989), and *Foliage Area Index*, FAI (Welles and Norman, 1991). Chen and Black (1992) used the term *effective LAI*, \( L_e \), to describe LAI estimates derived optically. [4]

\textsuperscript{16} James (aka Jacques, Jaques, Jakob, Jacob) Bernoulli (1654-1705) was first of three generations of perhaps the most famous mathematical family of all time. There were 8-12 Bernoulli mathematicians. Confusion arises as the same given names were used in successive generations. There were two Jameses — the present subject being the first; three Johanns — the first was James’ brother; and, two Nicolauses (not counting James’ father). [5, 6] He developed the binomial theorem around 1700 [4] but it was published after his death by Nicolaus (I) (a nephew to James and Johann(I)) in 1713 [5, 6, 7]. The well-known *Bernoulli’s equation* (c. 1733) used in hydrodynamics is named for Daniel Bernoulli (1700-1782) James’ nephew. [6]
If the probe is advanced and the trial’s outcome has no influence on another trial’s outcome (even at the same location), the trials are independent—a necessary condition. Also as the vegetation is uniformly distributed, the probability of a success of any trial is a constant and equal to \( p \). For any single trial, the probability of not contacting vegetation (failure) is \( q \) such that

\[
p + q = 1.
\]

(4)

For a population with binomial probably of success \( p \), the probability of \( n \) successes,

\[
p(n) = \binom{N}{n} p^n (1 - p)^{N-n}
\]

(5)

where, \( \binom{N}{n} = N!/(n!(N-n)) \) is the binomial coefficient.\(^{17}\)

For large \( N \), calculating tabular data for this distribution was an arduous task for Bernoulli and Abraham de Moivre; but, de Moivre, using the then new calculus, showed that for \( p = 0.5 \), the discrete binomial distribution closely approximated the continuous normal density function as the number of trials, \( N \), increased. [8]

The number of trials, \( N \), needed to confidently assume a normal distribution depends on the value of \( p \). The minimum such value of \( N \) occurs when \( p = 0.5 \) (binomial distribution is bell-shaped like the normal distribution). When \( p < 0.5 \) (binomial distribution is skewed), a normal approximation can conservatively be assumed when the mean \( \mu = Np > 15 \). But, in many applications, even for large \( N \), \( Np \ll 15 \) and the binomial distribution remains overtly skewed. So, Siméon-Denis Poisson (1837) developed a different approximation by working out the limiting form of the binomial distribution as \( N \) tends to infinity and \( p \), in concert, tends to zero (small probability of success of each

\(^{17}\) The binomial coefficient is also the \( n^{th} \) term of the \( N^{th} \) row of Pascal’s triangle (1654), where the first row is \textit{row zero} and the first term of each row is the zeroth term.
trial) such that $\mu = Np = \text{constant}$ or $p = \mu / N$.[2, 7] The development of the discrete Poisson distribution from the binomial distribution is

$$P(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$$= \frac{N!}{n!(N-n)!} \left( \frac{\mu}{N} \right)^n \left( 1 - \frac{\mu}{N} \right)^{N-n}$$

$$= \frac{N(N-1)(N-2)\cdots(N-n+1)(N-n)!}{N^n (N-n)!} \left( \frac{\mu^n}{n!} \right) \left( 1 - \frac{\mu}{N} \right)^{N-n}$$

$$= \left[ \frac{N}{N} \cdot \frac{(N-1)}{N} \cdot \frac{(N-2)}{N} \cdots \frac{(N-n+1)}{N} \right] \left( \frac{\mu^n}{n!} \right) \left( 1 - \frac{\mu}{N} \right)^{N-n}$$

If $N \to \infty$ (with $n$ and $\mu$ fixed), the terms $[(N-1)/N]-[(N-2)/N]\cdots-[(N-n+1)/N]$ will all tend to 1 and the rightmost term will tend to $e^{-\mu}$ according to the polynomial definition of $e^{-\mu}$, as

$$\lim_{N \to \infty} \left( 1 - \frac{\mu}{N} \right)^{N-n} = \left( \frac{1 - \frac{\mu}{N}}{1 - \frac{\mu}{N}} \right)^n = e^{-\mu}$$

Hence, the probability of $n$ successes in an unspecified, large number of Bernoulli trials is [2]

$$P(n) = \frac{\mu^n}{n!} e^{-\mu}; \quad n = 0, 1, 2, \cdots \quad (6)$$

This depends on the population mean, $\mu = Np$, the expected number of successes; but, not on the individual terms of $N$ (sample size) and $p$ (fixed probability of success).

![Figure 2–Typical, discrete Poisson distribution. Note the low probabilities on the vertical axis. As $N$ increases, these probabilities get even smaller.](image)

As an example, from a photo of $10 \times 10$ pixels one determines that $p = 0.7$ and constructs the bar plot of the discrete Poisson distribution shown in Figure 2. Note the
small probability of each event even though the plot is symmetrical about $\mu = Np = 70$. These low probabilities can be explained by the following turbidity model.

### A.3 Turbidity model

Assume a medium attenuates light passage by turbidity—e.g., the assumed homogeneous\(^{18}\) vegetal matter within a canopy where each successive lower leaf offers its shade as in Figure 3a or the fine, homogeneous droplet distribution in the cloud of Figure 3b. Now divide the vertical depth, $h$, into differential elements of depth $dz$, such that the probability of a photon being absorbed over distance $dz$ is very small and the possibility of two photons being intercepted is virtually zero. Thus, an individual Bernoulli trial is whether or not one photon is absorbed in its transit through distance $dz$ and the probability that one photon is absorbed, $p$, in each trial is extremely small. Then over the entire depth, $h$, the probability of $n$ interceptions ($n \geq 1$) is also very small. Now, consider the probability of a photon passing through the full depth, $h$, with no extinction. This is the probability of failure ($n$ held constant at $n = 0$ contacts), $P(0)$. Applying Equation 6, this is

\[
P(0) = e^{-\mu} = e^{-Np}
\]  

(7)

![Figure 3](image.png)

**Figure 3**—Light attenuation (extinction) in dilute media: (a) due to vegetation, (b) due to droplets in a cloud, and (c) due to exponential modeling of Beer-Lambert law.

\(^{18}\) It should be apparent that if the leaves or droplets were clumped in some fashion that more light would pass unobstructed. Also the obstructing objects are assumed to be dilute like tea which transmits some light as to heavy, Middle-Eastern coffee which passes no light and the bottom of the cup (or soil in our vegetation example) cannot be seen. Thus, percent cover must be less than 100% \(^{9}\).
For some reference volume of droplets, $V$, of height $h$, the plan area is $A = V/h$. If the concentration of droplets is $c$, then the total number of droplets is $N = cV$. Finally, the probability of a photon being absorbed in each *differential Bernoulli trial* is $p = a/A$; where $a$ is the cross-sectional area of a droplet. Combining these yields \[ P(0) = e^{-Np} = e^{-\left( cA h \right) / \left( a / A \right)} = e^{-ach}. \] (8)

So back to developing a PAI model. Reference [10] discusses a *suggested* model by Monsi and Saeki (1953, in German) for determining LAI (or PAI) and the model’s weaknesses when sunlight is not from directly overhead due to seasons, time of day and latitude. Nonetheless, the model has been used by many [3, 11]. It is based on the fact that the Poisson probability of light transmission (seeing soil), Equation 8, is of the same exponential form as the Beer-Lambert$^{19}$ extinction law which is discussed next.

**A.4 Beer-Lambert exponential extinction law**

The *Beer-Lambert exponential extinction law* states that the light intensity, $I$, within the turbid medium decreases exponentially with distance from an initial intensity, $I_o$ (see Figure 3c)

\[ I = I_o e^{-acz} \] (9)

where,  
$I_o$ – incident light intensity,  
$I$ – light intensity after passing through distance $z$,  
$z$ – distance that the light travels through the medium,  
$c$ – absorbing species concentration (turbidity particles),  
$a$ – absorbing species absorption coefficient.

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$^{19}$ The *Beer-Lambert law*, also known as *Beer's law* or the *Beer-Lambert-Bouguer law* is an empirical equation in optics relating the absorption (extinction) of light to the properties of the material the light is traveling through. It was independently discovered (in various forms) by Pierre Bouguer in 1729, Johann Heinrich Lambert in 1760 and August Beer in 1852. [12] “The Deeper the Glass, the Darker the Brew, the Less of the Incident Light that Gets Through” anon. [13]
The units of $c$ depend on the medium; for liquids, mole fraction (dimensionless) often represents concentration and thus $a$ has units of $(\text{length})^{-1}$. For gases, density often represents the concentration and $a$ becomes an absorption cross-sectional area with units of $(\text{length})^2$.

[12] It is the gas form of the Beer-Lambert law that strongly resembles Equation 8. Now apply the definition of PAI to the variables used in this equation

$$PAI = \frac{\text{Total Plant Area (Obstacles)}}{\text{Area of Quadrat}} = \left(\frac{\text{Total Number of Obstacles in V}}{\text{Area of Each Obstacle}}\right) \left(\frac{\text{Area of Reference Volume}}{A}\right) = \alpha \chi$$

(10)

A.5 Relating $p$ and PAI

In summary, relations have been established that relate the following:

- The binomial approximation of viewing soil in terms of percent cover, $q = 1 - p$,
- The Poisson approximation of viewing soil in terms of PAI, $P(0) = e^{-PAI}$, and
- The Poisson approximation is identical in form to the Beer-Lambert law of extinction.

Thus, one can write

$$1 - p = e^{-K(PAI)}$$

(11)

Here, an extinction coefficient, $K$, has been added. This will be discussed next.

A.6 Extinction coefficient

The extinction coefficient reflects Lambert’s cosine law$^{20}$ which says that if a plant part is tilted relative to the viewing (or illumination) angle, more light may get through. To elaborate, imagine a plant with leaves radiating in all compass directions, $\theta$, as shown in Figure 4a. Further, assume that the individual leaves are all inclined to the horizontal at a

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$^{20}$Lambert’s cosine law is the statement that the radiance of certain idealized optical sources is directly proportional to the cosine of the angle—with respect to the direction of maximum radiance—from which the source is viewed. [12]
Figure 4—Development of extinction coefficient

Foliage angle, $\alpha$, as shown in Figure 4b. Then, if an individual leaf has area $A$, the area viewed from above is $A \cos \alpha$. It should be apparent that in Equation 11, both the $p$ and
$K(PAI)$ are functions of $\alpha$. Similarly, when viewed from above, the viewed plan-area index is $PAI \cos \alpha$. Thus, the extinction coefficient is $K = \cos \alpha$. As vegetation may not necessarily be viewed or illuminated from above, a more general relation is needed.

The angle at which the vegetation is viewed or illuminated, measured from the horizontal, is called the quadrat angle, $\beta$ (see Figure 4b). For a random leaf with compass orientation $\theta$, leaf inclination $\alpha$, and view angle $\beta$, the apparent leaf area, $A_\beta$, is the actual leaf area, $A$, times the scalar product of the unit normal to the leaf and a unit normal parallel to the viewing direction. The direction cosines of these unit vectors are developed in Figures 4c through 4e. Thus, the apparent area is

$$A_\beta = A \left( \sin \alpha \sin \theta \hat{i} - \sin \alpha \cos \theta \hat{j} + \cos \alpha \hat{k} \right) \cdot \left( 0 \hat{i} - \cos \beta \hat{j} + \sin \beta \hat{k} \right)$$

$$= A \left( \sin \beta \cos \alpha - \sin \alpha \cos \beta \cos \theta \right)$$

(12)

Substituting $PAI$ for the area terms, Equation 12 can be applied to the computation of $PAI$ if each leaf had the same compass orientation. As the leaves are assumed uniformly directed to all compass points, the average value of

$$\frac{PAI_\beta}{PAI} = (\sin \beta \cos \alpha - \sin \alpha \cos \beta \cos \theta) \equiv g(\theta)$$

(13)

is desired. By symmetry, such is generally obtained by integrating $g(\theta)$ from 0 to $\pi$ and dividing by $\pi$. However, two different cases must be examined. Figure 4b is drawn such that the viewer or illumination always sees the top sides of the leaves. But, for $\beta < \alpha$, only the undersides of the leftmost leaves will be seen. For such leaves, Equation 8 yields negative values as the unit normal vectors of these leaves do not point from the undersides of the leaves. The value of $\theta$ at which the view becomes that of the foliage undersides, $\theta_o$, is
\[
\frac{\sin \beta \cos \alpha}{\cos \beta \sin \alpha} = \cos \theta_o \quad \text{or} \quad \theta_o = \cos^{-1}\left(\frac{\tan \beta}{\tan \alpha}\right)
\]  

(14)

Using the notation

\[
\left[h(\theta)\right]_0^\theta = \int_{\theta_1}^{\theta_2} g(\theta) \, d\theta = \frac{\left[\sin \beta \cos \alpha \theta - \sin \alpha \cos \beta \sin \theta\right]_0^\theta}{\theta_2 - \theta_1}
\]  

(15)

the two cases are

Case I, \( \alpha \leq \beta \):

\[\frac{PAI_\beta}{PAI} = [h(\theta)]_0^\theta = \sin \beta \cos \alpha \]  

(16)

Case II, \( \alpha > \beta \):

\[\frac{PAI_\beta}{PAI} = -\left[h(\theta)]_0^\theta \right] = \frac{2}{\pi} \sin \alpha \cos \beta \sin \theta_o + \sin \beta \cos \alpha \left[1 - \frac{\theta_o}{\pi/2}\right]
\]  

(17)

Finally, it is noted that Equation 11 is based on a downward view and may be written as

\[1 - p = e^{-K(PAI)} = e^{-PAI_{\beta=\pi/2}}
\]  

(18)

Figure 4b relates that \( PAI_\beta = PAI_{\beta=\pi/2} = \sin \beta \) so that

\[K = \frac{PAI_{\beta=\pi/2}}{PAI} = \frac{PAI_\beta}{PAI \sin \beta}
\]  

(19)

Thus, for Case I, \( \alpha \leq \beta \):

\[K = \frac{PAI_\beta}{PAI \sin \beta} = \frac{\sin \beta \cos \alpha}{\sin \beta} = \cos \alpha
\]  

(20)

as determined early on. And, for Case II, \( \alpha > \beta \):

\[K = \frac{PAI_\beta}{PAI \sin \beta} = \frac{2 \sin \alpha}{\pi \tan \beta} \sin \theta_o + \cos \alpha \left[1 - \frac{\theta_o}{\pi/2}\right]
\]  

(21)

For various values of \( \beta \) and \( 0 \leq \alpha \leq \pi/2 \), \( K \) plots as shown in Figure 5.
So, the theory of estimating PAI from percent cover is straight-forward when the vegetation is uniform, such as in a dense row crop, and its leaf (plant) angle distribution (LAD) is known. [11]

The extinction coefficient (or alternately the Leaf Angle Distribution) can be determined

- By meticulously dissecting and measuring the angles of vegetal parts relative to the plant axis,
- By looking up in data bases such as Reference 14,
- By assuming “the leaf angles are random, and $K = 0.5$ for a nadir view. This is a good estimate for plants whose leaves are individually at specific angles, but which average to all directions.” [11]

![Figure 5](image.png)

**Figure 5**—(After [10]) Extinction coefficient, $K$, as a function of penetration angle, $\beta$, and leaf angle distribution, $\alpha$.

**A.7 Vegetation density, $VEG_D$.**

The final vegetation index for consideration is vegetation density, $Veg_d$, which is akin to $LAI$ but measured in a horizontal direction. This term is the *apparent foliage denseness*
defined by Warren Wilson [15] except all plant parts are considered—not just the leaves.

Using a set of probes advanced horizontally a distance $D$ from a specified compass direction, $V_{egd}$ is defined as [1]

$$V_{egd} = \frac{\text{Number of Hits with Vegetation}}{\text{Number of Points (Probes)}} \times \left(\frac{1}{D}\right) = \frac{m}{D} \quad (22)$$

where, $m =$ average number of contacts per probe.

Above, $PAI$ was determined by advancing probes vertically through the entire plant height, $h$. Comparing the above equation with Equation 1 for $PAI$, it should be obvious that $V_{egd}$ is simply a horizontal $PAI$ per distance $D$. Alternately, $PAI$ is a vertical $V_{egd} \times h$. Both $PAI/h$ and $V_{egd}$ have units of one-sided leaf area per volume and this term is often used.

If one redefines the probability of seeing soil as the probability of seeing some reference surface, then $p$ is the same regardless of the direction of view—if the vegetation is homogeneous. Thus, $V_{egd}$ can be determined with a digital camera using a back-lit, vertical, white, translucent sheet of plastic at a known distance, $D$, into the vegetation.

More than one depth of vegetation could be used to test the theory and see if similar values of $V_{egd}$ are determined. Just as some soil must be visible in determining percent cover and plant-area index by the dilute turbidity model, so must some back lighting be visible to the camera in order for this method to work.

As an aside, FIRETEC uses a radiation heat transfer model to account for the radiation from the fire preheating the unburnt vegetation downwind of the fire. An optical density of vegetation for this radiation transmission can be determined from the calculation of $V_{egd}$, using a relation of the form of Equation 11, i.e., a percent of vegetation.
A.8 Nonhomogeneous vegetation

The discussion so far of variables describing vegetation density has assumed that the vegetation is *homogeneous*. However, unless one is considering dense row crops such as soybeans, most wildland vegetation will have gaps in the vegetation to the extent that only tussocks of grass may exist in an otherwise barren expanse. How does one account for these gaps?

First one can define the *gap fraction*. This is the $q$ term of Equation 4 such that

$$\left( \frac{\text{percent cover}}{p} \right) + \left( \frac{\text{gap fraction}}{q} \right) = 1 \quad (23)$$

Another way to investigate this is using a *gap size distribution* as outlined in Reference 16. Simply explained, imagine transiting a distance $L$ below a stand of foliage with an instrument oriented to view whatever is directly overhead. That view can *see* either foliage or sky (gaps between foliage elements) and that this instrument can determine the widths (along the transit line) of the gaps. Then statistically distribute the gap sizes.

This requires a measure of the average leaf width. The method is good when there is little variance in the gap size distribution regardless of the particular transit chosen. Even so, the averaging of many transits in varying directions is needed.

Finally, the use of fractals and lacunarity as discussed in the next appendix may prove useful. The gap size distribution above is developed along the same lines as lacunarity but makes no reference to it.
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Appendix B—Fractals and Lacunarity

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

The 19th-century mathematicians may have been lacking in imagination, but Nature was not.

... imitate Nature in order to guess its laws ...

—Benoît B. Mandelbrot [1]

In the 1993 play Arcadia by Tom Stoppard, ... the mathematician-protagonist, Thomasina, tells her young teacher Septimus—

Thomasina: Every week I plot your equations dot for dot, x’s against y’s in all manner of algebraic relations, and every week they draw themselves as commonplace geometry, as if the world of forms were nothing but arcs and angles. God’s truth, Septimus, if there is an equation for a curve like a bell, there must be an equation for one like a bluebell, and if a bluebell, why not a rose? Do we believe nature is written in numbers?

Septimus: We do.

Thomasina: Then why do your shapes describe only the shapes of manufacture?

Septimus: I do not know.

Thomasina: Armed thus, God could only make a cabinet. [2]

To the reader: In attempting to learn the material presented here, this person was confronted with mathematical terminology which was not well explained—too many circular definitions and too much reliance on notation. Thus, in the spirit of the classic Mechanical Vibrations (1934) by the consummate teacher J.P. den Hartog (1901-1989) this appendix was composed to explain concepts one already has a feeling for while minimizing mathematical notation. In retrospect, there is more material here than the reader needs; but, in trying to understand the mathematicians, once a nugget of understanding was unearthed this person was not about to set it aside. [3]
The word geometry comes from the Greek γεωμετρία (geo = earth, metria = measure). Yet, one can not draw a realistic mountain using the shapes of Euclidian (i.e., all standard) geometry. Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct length scales of natural patterns is for all practical purposes infinite. The existence of these patterns challenges one to study those forms that Euclid leaves aside as amorphous (formless, lacking organization). [1]

Most of nature’s irregular and fragmented patterns can be described by a family of shapes called fractals. The most useful fractals involve chance and both their regularities and their irregularities are statistical. Among the various applications of fractals, landscape renderings can now be created which include mountains, clouds, vegetation, and their reflections from lakes—thus beginning to fulfill the promise of the term geometry. A fractal approach is often times both effective and natural allowing the investigator to perceive the hidden order in the seemingly disordered, the plan in the unplanned, or the regular pattern in the irregularity and roughness of nature. Yet, it is not a panacea. [1]

Polish-born, French-American mathematician Benoît B. Mandelbrot (referred here as BBM, 1924– ), Benoît is pronounced as ben-wa, brot rhymes with ought) is considered the father of fractal geometry which, loosely speaking, is a way of measuring the roughness of an object or phenomenon. In his 1975 book Les objets fractals, he coined the term fractal (< Latin fractus, meaning broken, formed of irregular fragments) to bring together under one

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21 Fractals have been use to investigate turbulent flow, data transmission line noise, stock prices, metal fracture surface roughness, rugged coastlines, river patterns and yearly flows, brain waves during epileptic seizures, seismic tremors, distribution of galaxies, anthropology, Brownian motion, precipitation and pollution patterns, oil field reserves, and ice sheets. [4, 5]

22 Phonetic spellings in this appendix were found in References 6, 7, and 8.

23 The word algebra comes to us from the Arabic جبر (al-jabr) meaning the reunion of broken parts, from the verb جبر (jabara), to reunite, bind together. Thus, algebra and fractal are etymological opposites. Because of the Moorish influence on the Spanish language, Miguel Cervantes (1547-1616) in Don Quixote (II, Ch. 15), speaks of: “an algebrista (bonesetter) who attended to the luckless Sansón.” [1, 11]
heading a large class of objects that have influenced the development of modern mathematics and have precipitated from his own studies into natural and social phenomenon. He also introduced the terms \textit{fractal dimension} and \textit{self-similarity}. \cite{BBM}

BBM says that: \textit{an interest in the history of ideas (which the usual practice of science destroys) is good for the scientist’s soul.} This is based on his experience that after his own works in related areas were complete but his interests were not shared by anyone that he rejoiced in finding similar concerns expressed by some earlier investigators. Thus, the material below is presented from an historical perspective. \cite{BBM}

\section*{B.1 What is a fractal?}

In order to better understand the development of the various sections of this appendix, the reader needs to have at least \textit{a feeling} for what a fractal is. Since this overall work deals with plants, the images in Figure 1 have been chosen to demonstrate fractals. Thomasina said: \textit{there must be an equation for ... a rose.} The fractal in Figure 1a resembles an arrangement of roses. Taking a closer look, one will notice that the smaller clusters of flowers duplicate the overall arrangement—only scaled down. This is a basic feature of fractals—duplication at all scales of magnification. Fractal geometry is based on this ubiquitous \textit{scale invariance} where the object does not look any different when one magnifies it. If the smaller units scale the same in all directions, the shape is said to be \textit{self-similar}; if

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fractalPlants.jpg}
\caption{Fractal Plants. (a) \textit{Red Dahlia in Blue}, by Arend Nijdam, (b) A fern design by Barnsley using Iterated Function Systems (IFS). Each successively smaller element is as scaled down version of the entire plant. http://www.geocities.com/ffg/pl/flow15gr.jpg http://www2.umassd.edu/temath/TEMATH2/About/GraphOfBarnsleyFernBig.html}
\end{figure}
different scalings are used in different directions, the shape is said to be *self-affine*. [9]

In Figure 1b, the similarity of pattern is more apparent. The overall fern frond has many leaflets (the red and blue parts) which in this construction are identical to the frond—only scaled down. Closer examination reveals that the pinnules (subleaflets) are just scaled down images of the leaflets and the overall frond. [10]

Both of these fractals are generated mathematically. In nature, the scaled-down similarity is not quite as apparent and one must compare scaled-down views of an object using the statistical properties of chance. For example, over some range of magnifications, looking more closely at a rock or a coastline does not reveal a simpler picture, but rather more of the same *kind* of detail. A fractal is an object that doesn’t look any different when you magnify it and describes the roughness or distribution of an object. [9]

When a study involves some type of suspected scale invariance, a new dimension—beyond Euclidean dimensions—is needed. This dimension is called the *fractal dimension* and allows the researcher to perceive the order in the seemingly unordered. The term fractal dimension will be used in this work before it is fully defined. If an object is a fractal, its fractal dimension will be greater than its Euclidean dimension and usually noninteger. [1]

### B.2 Modern mathematics

Classical mathematics has its roots in Euclid’s geometry and Newton’s calculus. Modern mathematics burst forth during the last 25 years of the 1800s when functions were found that did not fit the patterns of Euclid and Newton. Some of these functions were even contrived to show the limitations of existing mathematics. [1]

In Section B.3 of this appendix, the major mathematical events of this period of evolvement which influenced the concept of fractal geometry will be highlighted. These
events were greatly influenced by (and, sometimes were the cause of) other co-emerging
concepts. Thus, the material of this section will address those concepts and terminology that
the reader needs to grasp before preceding to Section B.3.

Among the various fields of mathematics, are set theory, analysis, and topology. The
reader will soon realize that these fields greatly overlap. However, an attempt will be made
to sort the concepts to be discussed into their own bins. [7]

B.2.1 Set theory

A set can be thought of as any unordered list or collection of distinct objects (called
elements or members) considered as a whole. These objects can be anything: real numbers,
people, letters of the alphabet, points on a line, integrable functions, *et-set-tera*. The object of
set theory is to investigate the properties of sets from the most general point of view;
generality (independence of the nature of the elements comprising a set) is an essential aspect
of set theory. Though a simple idea, set theory is a most important and fundamental
foundation concept upon which nearly all of modern mathematics can be derived. [7, 12]

Set theory did not go through a slow evolution of ideas until one or more individuals
(often almost simultaneously) produced a major discovery in an inspirational flash. It is
the creation of German mathematician Georg Cantor (*kahn-tawr*, 1845-1918) alone.
Cantor’s work changed the whole course of mathematics and put set theory in a proper
mathematical basis. [13]

In 1869, Cantor moved from number-theory studies to study trigonometric (Fourier)
series. He had been challenged by a senior colleague to prove a problem on the uniqueness of
representing a function as a trigonometric series—he did. After publishing controversial
theories on sets in earlier papers, he birthed set theory in a paper in 1874. From 1879 to 1884,
Cantor published a six-part treatise on point-set theory. His studies proved that the set of real numbers was uncountable and contains irrational values. For twenty years he strove to prove that there were more points on a plane than a line. He and the world were shocked to find that this was not true. (Peano, discussed later, graphically proved this.) Cantor’s revelation opened the door to one-to-one mappings from a curve to a surface. Such mappings (functions) play an important role in set theory. These functions map the Set X onto Set Y so that for every two distinct elements of Set X correspond to two distinct elements of Set Y. It the inverse function, which maps the Set Y one-to-one onto the Set X, bicontinuity exists. [7, 12, 13, 14]

Although initially viewed with suspicion, set theory eventually became an indispensable basis in many branches of mathematics including the theory of measure (§B.2.2) and new mathematical disciplines such as topology (§B.2.3). [12]

Of primary importance to Section B.2.3 on topology is the concept of an open set (usually understood as part of the number line or plane). Cantor himself used the term closed set (ensemble fermé) in 1884 to mean a set that includes its own boundary. Thus, a closed interval (see Figure 2) will include all points on the interval including the end or boundary points and a closed disc will include all points on the disc as well as the boundary of the disc. At that time Cantor also says: in the next paper soon to come, I will show ... for not closed (non fermés) sets of points. Cantor never produced a next paper. [7, 11, 15]

Cantor hinted at the concept of open set as not closed and at least one text on set theory in 1906 used open set in that sense. Today, however, mathematicians use open set to mean the complement of the closed set as did Lebesgue (§B.2.2) in 1905. At that time, in a paper

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24 A boundary should not be confused with the concept of boundedness. The latter refers to a set being of finite size as to being bounded.
on analytically representable functions Lebesgue wrote: open (overts) sets, are those which are complementary to closed (fermés) sets. [11, 15]

So just what is meant by the complement? Let \( U \), the universal set (aka universe), be a set that is large enough to contain, in some sense, all the sets that one may wish to use. Say \( U \) is a plane infinite in two directions. Now, partition the plane into a closed Set \( A \) which is composed of all points in the interior of \( A \) as well as its boundary points and the Set \( B \) of all points exterior to \( A \). The Set \( B \) is the complement of \( A \) and does not include points on the common boundary between the sets and thus is an open set. [16]

Another type of point the reader may encounter in reading was described by Cantor in his 1872 paper *Über die Ausdehnung eines Satzes der Theorie der trigonometrischen Reihen* (*On the theory expansion of trigonometric series*). This is the *accumulation point* (*Häufungspunkt*). This point of accumulation is also called a *limit point*. [11]

This is best understood by example. The open interval \((0, 1)\) includes all real numbers between 0 and 1, except 0 and 1. However, for either endpoint, an infinite sequence may be defined that converges to it in the limit. For example, the sequence \(1/2, 1/4, \ldots, 1/2^n\)
converges to 0 as \( n \) tends to infinity. This means that one can choose a point in \((0, 1)\) within any small, positive distance from either 0 or 1; but, one cannot pick one exactly on the boundary of the interval. A limit point is any point on the boundary or interior of a set. Thus, the set of limit points is a closed set. This somewhat explains the adjective limit but this person has no understanding of the adjective accumulation. [17]

The term connectedness in set theory is used in the sense: all one piece. If it is not, it is disconnected or disjoint. A space (set) is said to be connected if it cannot be partitioned into two disjoint open sets. A set is open if it contains no point lying on its boundary; thus, in an informal sense, the fact that a space can be partitioned into disjoint open sets suggests that the boundary between the two sets has been removed from the space, splitting it into two separate pieces. [7]

Karl Weierstrass (vi-uhr-shtrass, German, 1815-1897) used connected open sets in \( n \)-space and called such a Gebiet (region). German Felix Hausdorff (howz-dohrf, 1868-1942) in 1914 took over Weierstrass’s term Gebiet, but changed it to mean an open set, and explicitly referred to Weierstrass’ earlier usage of the term to mean open, connected set. Both of these mathematicians as well as Cantor will be discussed in Section B.3. [11, 15]

**B.2.2 Measure**

Analysis is the generic name given to any branch of mathematics that depends upon the concepts of convergence and continuity. It studies closely related topics such as integration, differentiability, and transcendental functions. Analysis has its beginnings in the rigorous formulation of calculus. [7]

The concept of measure is a subfield of analysis. A measure is a function that assigns a number (e.g., a size, length, volume, or probability) to subsets in a set. Measures allow one to
compare sizes of subsets. Measure theory deals with systematic techniques for measuring complicated or irregular objects *when the measurements of simple objects are known in advance*. The simple objects of known measure might be thought of as *rulers* of various dimensions. A 1-\(D\) ruler is not unlike the ruler in one’s desk drawer. A 2-\(D\) ruler would be like a tile which could be overlaid on an area to be measured. A three-dimensional ruler would be like a brick—many of which would compose some volume. [7, 18, 19, 20]

In the time of Archimedes of Syracuse (287BC-212BC), the Greeks measured land area using the geometry of Euclid of Alexandria (325BC-265BC) to *pave* an irregular plot of unknown area with rectangles and triangles *whose areas could be determined*. These paving forms and the fact that *the total area equaled the sum of the areas of the individual paving areas* was possibly the beginning of measure theory. English natural philosopher Isaac Newton (1643-1727) used rectangular paving as shown in Figure 3a to approximate area under a function. The figure implies both *upper* and *lower* bounds to the rectangular approximation. If greater accuracy was needed, narrower rectangles were used to *cover*\(^{25}\) the region. It was known that the exact area was the limit as rectangles got narrower. This limit was called the *integral* of the function over the specified interval. [19]

French mathematician Augustin-Louis Cauchy (*koh-shee*, 1789-1857) in the 1820s clarified Newton’s notions of derivative and integral by basing them on the idea of a limit. Bernard Riemann (*ree-mahn*, 1826-1866) extended Cauchy’s work in the 1850s and the resulting integral bears his name. [19]

\(^{25}\) *covering*: (German *Belegung*, from the verb *Belegen*, to cover) was first used by Georg Cantor in his last works (1895-97) on set theory. [11]
A drawback of Riemann integration is that it embraces Euclidian coordinates (line, plane, etc.) so it is unable to handle arbitrary sets. For example, with the advent of Fourier series, it could not describe how and when one can take limits under the integral sign; i.e.,

\[
\lim_{k \to \infty} \sum_{k=0}^\infty \left[ \int f_k(x) \, dx \right]^2 = \int \left[ \lim_{k \to \infty} \sum_{k=0}^\infty f_k(x) \right] \, dx
\]

This drawback encouraged investigations that lead to measure, which is a central concept of modern mathematics. [7, 20]

Modern mathematical measure theory starts with Italian mathematician Giuseppe Peano\(^{26}\) (pee-ahn-oh, 1858-1932) in the 1880s. He considered area, in its own right, as distinct form integral; and defined inner and outer contents. Referring to Figure 3b, the inner content is the sum of the areas of rectangles inscribing (drawn within another figure) the shape. Depending on the number, sizes, and arrangement of these rectangles, this inner-content sum must be less than or equal to the

\[(a)\] If in any figure AacE, ... there be inscribed any number of parallelograms Ab, Be, Cd, &c., comprehended under equal bases AB, BC, CD, &c., ... Then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented in infinitum; I say, that the ultimate ratios which the inscribed figure AKBLCMD, the circumscribed figure AaBbCcDdE, and curvilinear figure AaE, will have to one another, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms K1, Lm, Mn, Do, ... But this rectangle, because its breadth AB is supposed diminished in infinitum, becomes less than any given space. And therefore the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate figure be ultimately equal to either. [Newton, I., Principia (tr., Andrew Motte, 1729), 1687; Posted by gravity@thevortex.com at http://members.tripod.com/~gravitee/]

\[(b)\] Newton’s wall paperings showing both maximum and minimum areas [from 19]; \[(c)\] Peano’s inner and outer content [after 19]; \[(d)\] Lebesgue tiles as rulers (pavers) of lines and surfaces [after 21]; \[(e)\] Carathéodory’s covering a line with balls; \[(f)\] Carathéodory’s development to show that measure is only dependent on outer measure. [19]

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\(^{26}\) It is worth noting that Peano introduced the symbol \(\in\) for is an element of in 1889. It comes from the first letter of the Greek word meaning is (eίναι). [13]
exact area. The maximum (or, when this does not exist, the generalized maximum or supremum\(^{27}\)) of all such paved areas constitutes an underestimate of the area. At the same time, Peano considered the totality of rectangular pavings which completely cover or circumscribe the area. The minimum (or where this does not exist the generalized minimum or infimum\(^{28}\)) of the area covered by such pavings constitutes an overestimate of the area. This he called the outer content. [19]

Peano used a finite number of rectangles. French mathematician Émile Borel (boo-rel, 1871-1956) used infinitely many non-overlapping rectangular pavings. He is credited with creating the theory of the measure of sets of points in 1894. Borel’s measure is not as robust as the measure created by his four-year-younger student Henri Lebesgue (leh-beg, 1875-1941). Like Riemann, Lebesgue was more interested in integration than in measurement. Lebesgue first set out his theory of measurable sets and measures on these sets in 1901. He constructed his measure with the goal of extending Riemann’s definition of integral. This he did the following year when he defined the theory of measurable functions and integrals on these functions. Both were published as part of his dissertation [Intégrale, longuer, aire (Integral, length, area)] in 1902. Indeed, Lebesgue integration did refurbish Riemann integration into a more-powerful, modern theory. Instead of being limited to partitioning the domain of integration into intervals, virtually any partition into measurable sets can be used. Its definition requires the notion of a measurable function to ensure that the function domain is partitioned into measurable sets. [7, 11, 17, 19, 20]

\(^{27}\) Supremum, pl. -ma (< Latin supremus, highest), abbr. sup. Same as inner measure of a set; i.e., the greatest lower bound of the measures of the closed sets contained in a given set. Whenever the supremum exists, its value is unique. [22, 23]

\(^{28}\) Infimum, pl. -ma (< Latin infimus, lowest), abbr. inf. Same as outer measure of a set; i.e., the least upper bound of the measures of the open sets containing the given set. Whenever the infimum exists, its value is unique. [22, 23]
The Lebesgue outer measure of a planar set is defined in the same way as Peano’s outer content; except, that infinite pavings are used as well as finite ones. This increases the number of paved areas under consideration and so the infimum of these areas has to be smaller than the Peano’s outer content. [19]

Lebesgue hoped that his outer measure would serve as a true area measure; but, this hope needed justification. Mainly, he had to show that the whole measure was always the sum of the measures of the parts. Lebesgue dealt with this by creating a new notion of inner measure. This concept is now archaic and is not analogous to Peano’s inner content. Lebesgue claimed that a set was (Lebesgue) measurable if the outer and inner measures were equal. Thus, the actual measure equaled the common value of outer and inner measures. He also established the countable additivity of outer measure on nonoverlapping measurable sets; i.e., he showed that the whole is equal to the sum of the parts for measurable sets. [19]

Lebesgue only constructed his measure for curves (1-D); a generalized length function that used intervals as paving stones. His approach was easily extended to surfaces and volumes using rectangles and rectangular parallelepipeds (n-dimensional tiles or bricks aligned with the coordinate axis)—see Figure 3c. Tile intersections occurred at most at their common boundary. [19]

Lebesgue’s work produced the existence of sets of zero measure. The Lebesgue measures of a closed interval \([a, b]\) and of an open interval \((a, b)\) are the same; i.e., the length \((b - a)\). Thus the difference in the length measures of the two sets is a Lebesgue measure of zero—the length of a point or a countable set of points. [7]

Just as a zero-dimensional point has no length when using a one-dimensional tile, a surface has no volume when being measured with a three-dimensional ruler. If the
surface were measured using two-dimensional tiles, a nonzero measure would be obtained. This will be discussed further under Hausdorff measure. [17]

That the general notion of volume or magnitude is indispensable in investigations on the dimensions of continuous sets occurred to Cantor in passing. Given the difficulty of the problem, Lebesgue doubts that Cantor could reach any significant result. [1]

Constantin Carathéodory (kar-a-thee-a-dor-ee, 1873-1950), who was German of Greek descent, took up the work. In 1914, he published On the Linear Measure of Point Sets—a generalization of the Concept of Length. In this, he addresses Lebesgue’s work and extends the theory to generalized arc length in other dimensions. Carathéodory changed Lebesgue’s use of tiles, which imply coordinate axes, to the use of discs and strives not to use the fact that the object to be measured is a standard shape of known dimension imbedded in Euclidian space. The generalization of how to define \( p \)-dimensional measure for a set in \( q \)-dimensional space (where \( p \) is integer) is then easily defined in the same way. [1, 24]

When a planar shape imbedded in Euclidian space is covered by discs of radius \( r \), it is actually covered by balls\(^{29}\) of which the discs are equators. So, to avoid prejudging the fact that an object is planar, it suffices to cover it by balls as to discs. Figure 3d shows a line being covered by balls. When the object is indeed a surface, one obtains its approximate content by adding expressions of the form \( \pi r^2 \) corresponding to all the covering balls. More generally in Euclidean space, a \( D \)-dimensional standard form (ruler) of characteristic size \( r \) requires one to sum expressions of the form

\[
h(r) = \gamma(D) r^D,
\]

where, \( \gamma(D) \) is the measure of an \( D \)-dimensional ball of unit radius and is defined as\(^{30}\)

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\(^{29}\) ball: a solid object the boundary of which is a sphere. [7]

\(^{30}\) This relation is in Hausdorff’s 1919 paper (§3.6); but, who is credited for it?
where, the gamma function\(^{31}\), \(\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt\), has identity \(\Gamma(\frac{1}{2}) = \sqrt{\pi}\) and recursion property \(\Gamma(x + 1) = x\Gamma(x)\). Thus, for Euclidean values of \(D\), one can construct Table 1.\(^{32}\) On this basis Carathéodory, extends the ideas of length or area to some nonstandard shapes. [1]

Lebesgue said a set is measurable only if its inner and outer measures were equal, the main advantage of Carathéodory’s method is that it is independent of the concept of an inner measure. Consider Figure 3e, where Set \(S\) is to be measured. Consider also Set \(D\) which is partitioned by \(S\) into \(D_1 (D \cap S)\) and \(D_2 (D \setminus S\) or \(D\) minus \(S\)). Carathéodory showed that if it is always true, regardless of the choice of \(D\), that the measure of \(D\) is the sum of the (outer) measures of \(D_1\) and \(D_2\) [written as \(\mu(D) = \mu(D_1) + \mu(D_2)\)], then \(S\) is measurable. It is an abstract definition, and it was created for the purpose of forcing the whole to be equal to the some of the parts. \(D\) might be a disc covering a point or surface smaller than \(D\). [19, 24]

All paving stones (rulers) are measurable (have a known measure)—this is a theorem in measure theory. [19]

\(\gamma(D) = \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{D}{2}\right)}\right]^D = \frac{\pi^{D/2}}{\Gamma\left(1 + \frac{D}{2}\right)}\)

<table>
<thead>
<tr>
<th>Table 1—Euclidean Rulers</th>
<th>[1]</th>
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<tbody>
<tr>
<td>(D)</td>
<td>(\gamma(D))</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>(\pi)</td>
</tr>
<tr>
<td>3</td>
<td>(4\pi/3)</td>
</tr>
</tbody>
</table>

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\(^{31}\) gamma function: introduced during the years 1729 and 1730 by the Swiss mathematician Leonhard Euler whose goal was to generalize the factorial to noninteger values. [25]

\(^{32}\) For completeness, the null set, \(\emptyset\), has topological dimension \(D_T = -1\). Thus, \(\gamma(-1) = 1/\pi\) and \(h(r) = 1/(\pi r)\).
B.2.3 Topology

The root of topology\(^{33}\) (< Greek, τοπος, place or position; λογος, study) is in Euclidean geometry. German philosopher Gottfried Leibniz (\(\text{lyb-nits}, 1646-1716\)) is the first to refer to topology as analysis situs (Latin, analysis of place) in a letter to Dutch natural philosopher Christiaan Huygens (\(\text{hy-genz}, 1629-1695\)) in 1679, where he says: I believe that we lack another analysis [beyond analytic geometry] properly geometric or linear which expresses location directly as algebra expresses magnitude. [26]

In 1736, Swiss mathematician Leonhard Euler (\(\text{oy-ler}, 1707-1783\)) presented a paper on the Seven Bridges of Königsberg (Kaliningrad) that is regarded as one of the first topological results. The English translation of the title is: The solution of a problem relating to the geometry of position—implying distance was not relevant. The motivating insight behind Euler’s study is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are connected together. [7, 11, 13]

The term topologie was introduced in German in 1847 by Johann Listing (1808-1882) in a paper entitled Preliminary Studies in Topology; however, he had been using the term for ten years as a replacement to analysis situs. In 1861, Listing published a much more important paper in which he described the Möbius band [four years before German August Möbius (\(\text{may-bee-uhs}, 1790-1868\))] and studied components of surfaces and connectivity. Listing examined connectivity although Riemann had studied it earlier in 1851 and in 1857. Listing came by his interest in topology through Johann Gauss (\(\text{gowss}, 1777-1855\)), who chose not to publish topological works. Topology as an English word is found in the February 1883 Nature which says: The term Topology was introduced by Listing to

\(^{33}\) Do not confuse topology with topography, which is the study of the shape and nature of terrain.
distinguish what may be called *qualitative* geometry from the ordinary geometry in which quantitative relations (size, angle, distance) chiefly are treated. [7, 11, 13]

From around 1925 until 1975 topology was the main growth area in mathematics. Today, its concepts exist in almost all areas of mathematics. The subject itself developed in several different directions and consists of several different branches having little in common. The two earliest branches are: [7]

1. *General* or *point-set* or *set-theoretical topology* which grew from Cantor’s (1874) theory of point sets. This defines and studies properties of mapping functions and arbitrary subsets of Euclidean space. [7, 12, 13]

2. *Algebraic topology* was founded about the same time by French engineer and mathematician Henri Poincaré (*pwahn-kah-ray*, 1854-1912). He has been called the *next universalist* after Gauss; *i.e.*, one at ease in understanding and in contributing to all branches of mathematics, both pure and applied. He contributed to fluid mechanics and

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**Figure 4**—Some Notables Mentioned in Section 2

<table>
<thead>
<tr>
<th>Name</th>
<th>Image</th>
<th>Image</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclid of Alexandria</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>Archimedes of Syracuse</td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
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<tr>
<td>Christiaan Huygens</td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
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<tr>
<td>Isaac Newton</td>
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<tr>
<td>Gottfried von Leibniz</td>
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<tr>
<td>Leonhard Euler</td>
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<td><img src="image12.png" alt="Image" /></td>
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<td>Augustin Louis Cauchy</td>
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<td><img src="image14.png" alt="Image" /></td>
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<tr>
<td>G. F. Bernhard Riemann</td>
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<td><img src="image16.png" alt="Image" /></td>
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<tr>
<td>Jules Henri Poincaré</td>
<td><img src="image17.png" alt="Image" /></td>
<td><img src="image18.png" alt="Image" /></td>
</tr>
<tr>
<td>L. E. J. Borel</td>
<td><img src="image19.png" alt="Image" /></td>
<td><img src="image20.png" alt="Image" /></td>
</tr>
<tr>
<td>Constantine Carathéodory</td>
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<td><img src="image22.png" alt="Image" /></td>
</tr>
<tr>
<td>Maurice René Fréchet</td>
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<td><img src="image24.png" alt="Image" /></td>
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<tr>
<td>Émile Borel</td>
<td><img src="image25.png" alt="Image" /></td>
<td><img src="image26.png" alt="Image" /></td>
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<tr>
<td>Henri Léon Lebesgue</td>
<td><img src="image27.png" alt="Image" /></td>
<td><img src="image28.png" alt="Image" /></td>
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<tr>
<td>Pavel Urysohn</td>
<td><img src="image29.png" alt="Image" /></td>
<td><img src="image30.png" alt="Image" /></td>
</tr>
<tr>
<td>Karl Menger</td>
<td><img src="image31.png" alt="Image" /></td>
<td><img src="image32.png" alt="Image" /></td>
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</tbody>
</table>
the special theory of relativity. He published *Analysis situs* in 1895 which was the first real systematic look at topology. He built his work on foundations developed by those already mentioned. Poincaré formed many of his ideas while studying ordinary differential equations which arose from a study of stability problems in celestial mechanics. His study of autonomous systems\(^{34}\) involved looking at the *totality of all solutions* rather than at particular trajectories as had been the case earlier. \([13, 27]\)

Algebraic topology is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to continuous (smooth) transformations that can be continuously undone (bicontinuity). These two processes (function and inverse function) are continuous in the sense that during each of them, *nearby* points at the start are still nearby at the end.\(^{35}\) Topology operates with more general concepts than analysis; differential properties of a given transformation are nonessential for topology, but bicontinuity is essential. Thus, topology is often suitable for the solution of problems to which analysis cannot give the answer. The generality of topological methods rests in both the generality of the assumptions concerning the transformations and the generality of the sets that are transformed. How much more general ought the spaces considered in topology be in order that they suffice for applications and yet,

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\(^{34}\) **autonomous systems**: If time is represented by \(t\), displacement by \(x\), velocity by \(v = \frac{dx}{dt} = \dot{x}\), and acceleration by \(a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}\); then, a differential equation or system of ordinary differential equations is said to be autonomous if it does not explicitly contain the independent variable \(t\). A second-order autonomous differential equation is of the form \(F(x, \dot{x}, \ddot{x}) = 0\). Applying the chain rule yields \(\dddot{x} = \ddot{v} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt} \dot{x} = \frac{dx}{dt} = \frac{dx}{dc} \frac{dc}{dt} = \frac{dx}{dc} \dot{c}\). The solution to such is independent of time at which the initial conditions are applied. \([22]\)

\(^{35}\) Cantor’s early mapping of curves onto surfaces were *not continuous* in that points that are near one another on a surface may be unthreaded into two points on a curve that are not close to one another. This is true of Peano’s graphic construction also. \([14]\)
because of undue generality, they do not become too artificial? The answer to this question depends on the aims which a given topological work is to serve. [7, 12, 22, 28, 29]

Algebraic topology is popularly known as rubber-sheet geometry where a rubber disc can be deformed into a triangle without cutting or gluing. In order to deal with such problems that do not rely on the exact shape of objects, one must be clear about just what properties these problems do rely on. From this need arises the notion of topological equivalence. Formally, two spaces are topologically equivalent (have the same topological property) if there is a homeomorphism between them. One homeomorphism is the bicontinuous stretching and bending into a new form. Another topological invariant is the property of a torus (doughnut) to separate a volume into two regions. If the doughnut is stretched into a coffee cup this property is retained. In this last example, both objects have a single hole. Algebraic topology has a great deal of mathematical machinery for studying different kinds of hole structures, and it gets the prefix algebraic since many hole structures are represented best by algebraic objects like groups and rings. Set-theoretic topology, in contrast, uses the concepts and theorems of set theory. [7, 12, 22]

French mathematician Maurice Fréchet (fra-chay, 1878-1973), introduced the concept of metric space in 1906. A metric (from Greek μέτρον, a measure) or distance function defines a distance between elements of a set. A set with a metric is termed a metric space. The metric space corresponding to ones intuitive understanding of space is 3-D Euclidean space—the Euclidean metric being a straight line between any two points of an object. [7]

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36 Homeomorphism should not be confused with homomorphism. The Greek prefix homeo- means similar or like, the prefix homo- means same or identical, and the Greek word morph means form, shape, or appearance. In abstract algebra, a homomorphism is a structure-preserving map between two algebraic structures (such as groups, rings, or vector spaces)—whatever that means. A homeomorphism is a permitted topological transformation. [7, 30]
In 1914, German mathematician Felix Hausdorff published his famous text *Grundzüge der Mengenlehre (Fundamentals of set theory)*\(^{37}\) which, building on the works of others, generalized Fréchet’s notion of metric space. In that paper, he coined the terms metric space and topological space. The idea of a metric allows one to apply analytical ideas to a variety of spaces. However, in using a metric on a space, often one is asking for too much information. It often happens that when one constructs new spaces by gluing things together, like a Möbius strip, having to provide a metric on them is often difficult, sometimes impossible and always irrelevant—metric spaces do not let one do everything one wants. To handle a situation like this needs a more general structure; and, this structure is a topological space. Thus, every metric space is a topological space; but, not all topological spaces are metric spaces. While metric spaces deal with the concept of distance, topological spaces merely consider nearness or closeness. This follows the general principle of topology to deal in general concepts. [7, 13]

**B.2.4 Dimension**

Dimension (<Latin *dimetiri*, to measure out) has been referred to above; yet, it needs to be discussed by itself. The concept of dimension, even though it dates from antiquity (it appears in Euclid’s *Elements*\(^{38}\)), was properly defined only in recent times. The dimension of an object describes the way in which it occupies space, and thus how its size might be quantified. [12]

Intuitively, the dimension of the space equals the number of real parameters necessary to describe different points in the space. This view of dimension was badly shaken with Cantor’s

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\(^{37}\) This work marks the emergence of point-set topology as a unique discipline. [24]

\(^{38}\) Euclid of Alexandria lived from 325BC to 265BC. No doubt many ancient societies used the concept of dimension. In *Sefer Yetzirah*, the first chapter of which is attributed to the Patriarch Abraham (c. 1812BC-1637BC, later chapters may have been written by others), dimension is mentioned as follows: Ten Sefirot (countings) of Nothingness (ineffable): Their measure is ten which have no end; A depth of beginning, A depth of end; A depth of good, A depth of evil; A depth of above, A depth of below; A depth of east, A depth of west; A depth of north, A depth of south. Here one recognizes Euclidean 3-space, the time dimension, as well as a metaphysical dimension. [30]
proof that there is a one-to-one correspondence between a curve and a surface and Peano’s construction of a space-filling curve (§B.3.3). Thus, the understanding of dimension was found wanting. [14]

The first general definition of dimension of a space $X$ was given in 1913 by L.E.J. Brouwer (brow-uhr, Dutch, 1881-1966) for metric spaces. In 1921, Pavel S. Urysohn (ooh-ree-sohn, Ukrainian, 1898-1924) and Karl Menger (Austrian-American, 1902-1985), independently of Brouwer and of each other, arrived at a similar definition. Both are defined for integer dimensions using inductive reasoning that separates a space into subspaces of smaller dimension. The Brouwer definition is called the large inductive dimension, $\text{Ind } X$, and the Uryshon-Menger definition is called the small inductive dimension, $\text{ind } X$. [23]

Earlier in 1911, a completely different approach to the concept of dimension originated from Lebesgue which made it possible to geometrize the concept of dimension. This can be best understood from Figure 5a where a two-dimensional square is covered by $n$-dimensional balls which appear as two-dimensional discs in the figure. If the discs (or the spheres in $n$-space) just touched at their boundaries, the square could not be completely

![Figure 5](image_url)

Figure 5—(a) Covering a square with discs refined such that at most three discs overlap. (b) An unrefined covering of the same square where the large red disc causes four discs to overlap in places. (c) Two coverings of a Koch curve where the smaller diameter discs are a refined cover of the covering by the larger discs. [after 14]
covered; so in order to fully cover the square, the discs must overlap. In some regions of the square shown, there is no overlapping of discs; while in other regions, parts of two or three discs overlap. Lebesgue\textsuperscript{39} said that if $n + 1$ (in this case 3) discs overlap, then the object being covered is $n$- (in this case 2-) dimensional; \textit{i.e.}, $\dim X = n$. The minimum value of the greatest number of overlapping is called the \textit{multiplicity} or \textit{order} of the covering. The diameters of the individual spheres may vary; but, after throwing away some spheres that are too large (or reducing their diameters), the minimum value of this greatest number of overlappings is one greater than the dimension of the object being covered. This theorem was first proved by Brouwer. [14, 23]

The method of reducing the diameters of the covering elements is called a \textit{refinement}. Referring to Figure 5c, a Koch curve (which is constructed as shown in Figure 6d) is first covered by the larger discs. A refinement of the first cover is the set of smaller discs such that all of the smaller discs lie within (inscribe) the larger discs. The idea is that the diameters of the second set are in some sense smaller/finer than those in the first set and provide a more detailed coverage. As the covering is further refined and the diameter of the largest disc in the set, $\varepsilon_0$, goes to zero, there is no covering of the curve with multiplicity greater than $n + 1$. Since the Koch curve is one dimensional, the multiplicity of the covering should be 2 after refinements; whereas the multiplicity of the square first considered is 3. In Figure 5b, the lower-center disc of the Figure 5a has been enlarged causing the number of discs overlapping some points in the square to be four. A refinement of this set is the set in Figure 5a. [14, 23]

\textsuperscript{39} Lebesgue, H., Sur la non-applicabilité de deux domaines appartenant à des espaces à $n$ et $n + p$ dimensions (On the non-applicability of two fields belonging to spaces of $n$ and $n + p$ dimensions), \textit{Math. Ann.}, 70, pp. 166-168, 1911. Lebesgue conjectured that $\dim = n$ for the \textit{n-dimensional cube $I^n$}. Brouwer was the first to prove this, as well as the stronger identity: $\dim I^n = \text{Ind } I^n = n$. Urysohn proved that for a space $X$, $\dim X = \text{ind } X = \text{Ind } X$ [23]
This dimension is called the \textit{Lebesgue covering dimension} or \textit{Lebesgue dimension} or \textit{covering dimension} or \textit{Čech-Lebesgue dimension}; and, as stated further on, is also called the \textit{topological dimension}, $\dim X = D_T$. A point or a \textit{countable} set of points has topological dimension, $D_T = 0$, a curve has $D_T = 1$, a surface has $D_T = 2$, and a volume has $D_T = 3$. The null (empty) set, $\emptyset$, is by definition of topological dimension $–1$. The dimension on any other space will be defined as one greater than the dimension of the object that could be used to completely separate any part of the first space from the rest. [9, 23, 29, 31]

Lebesgue used \textit{closed} covering elements in his \textbf{1911} definition of dimension as well as in his previous \textbf{1901} work on measure where his tiles overlapped at most at their edges. Yet, most (but not all) topology references use the notion of \textit{open} sets as fundamental elements. This is because it is natural to consider open coverings because the elements of such a covering contain complete information of the local structure (\textit{i.e.}, the topology) of the space. So when closed coverings are considered (as they often are) other information on the surrounding space must be provided. [23]

\textbf{B.3 Modern mathematic works leading to the concept of fractal dimension}

Classical mathematics has its roots in Euclidian geometric structures and Newton’s calculus. \textit{Modern} mathematics began with Russian-German mathematician Georg Cantor invented set theory (developed during the years of 1871-1884) and Italian Giuseppe Peano invented the \textit{space-filling} or \textit{area-filling} curve in 1890. Generally, the revolution was forced by the discovery of mathematical structures that did \textit{not} fit the patterns of Euclid and Newton. These new structures were regarded as \textit{pathological} and as \textit{a gallery of monsters}. These and other \textit{pathological} structures are presented in Figure 6. [1]
B.3.1 Weierstrass’ function

Fractal geometry was born belatedly of the crisis that started in Germany when du Bois-Reymond (German, 1831-1889) in 1875 reported on an *everywhere continuous* but *nowhere differentiable* function (see Figure 6a) constructed by Karl Weierstrass. The crisis lasted until about 1925. What makes this function significant is that it has uniform and infinite complexity at all levels of magnification. For this reason, the curve does not appear to smooth out as one zooms in; thus, no tangent can be equated to the graph at any one point and the function is not differentiable. The graph of this function would now be called a fractal. [1, 7]

Weierstrass’ function led French mathematician Charles Hermite (air-

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40 Weierstrass first constructed this in 1872. Bernard Bolzano (1781-1848) from what is now the Czech Republic devised a continuous, nowhere-differentiable function in 1834; but, his work remained unpublished until the 1920s. Riemann had suggested in 1861 that such a function could be found; but, his example failed to be nondifferentiable at all points. [12, 26]
meet, 1822-1901) to remark: *I turn aside with a shudder of horror from this lamentable plague of functions which have no derivatives.* [19]

**A.3.2 Cantor’s Set**

The original Cantor set of Figure 6b is formed by removing the center 1/3 of the closed interval [0, 1] and then successively removing the center 1/3 of the remaining intervals. The intervals removed are open intervals. As the number of remaining segments becomes uncountable (infinite in number), the segment lengths go to zero and the nowhere dense (totally dispersed) set of disconnected points on the interval [0, 1] that are not removed at any step in this infinitely recursive process is called a Cantor set or Cantor dust. The total length of open intervals removed is

\[
\frac{1}{3} + 2\frac{1}{3} + 4\frac{1}{3} + 8\frac{1}{3} + \cdots = \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} = \frac{1}{3} \left(\frac{1}{1-\frac{2}{3}}\right) = 1
\]

and the total length of the remaining dust is

\[
\lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.
\]

From the calculation, it may seem surprising that there would be anything left—after all, the sum of the lengths of the removed intervals is equal to the length of the original interval. The Cantor set contains no measurable intervals (Lebesgue measure zero). Yet, there are an infinite number of points left behind—these are the boundary points (limit points) of the closed intervals left behind. Said differently: The Cantor set is not empty and these points exist since the middle-1/3 intervals that were removed were open sets. As these limit points are uncountable but nowhere dense, their Lebesgue measure is 0; i.e., as each point has measure zero, the sum of the measures is also zero. [7, 22]
Cantor himself was led to developing this set by practical concerns about the set of points where a trigonometric series might fail to converge. The discovery did much to set him on the course for developing an abstract, general theory of infinite sets. [7]

As stated earlier, Cantor proved that a line and plane contained the same number of points. Cantor’s dust (dimension 0) also has an infinite number of points just as the original line segment did. Points have topological dimension zero; but, when their number becomes uncountable their fractal dimension is between 0 (the dimension of a point) and 1 (the dimension of a line) as will be discussed further on.

In 1958, BBM moved to the U.S. to take up a research position at IBM’s Thomas J. Watson Research Center. (He left France because of Bourbaki.41) Soon after joining IBM, he was

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41 Bourbaki (boor-bah-kee): France lost many young science and math scholars in the First World War. After the war, students had no previous generation for direct guidance. So in 1935, a youngish, mainly-French group of mathematicians began writing a series of mathematics texts under the collective allonym of Nicolas Bourbaki. They strove to found all of mathematics on set theory and the utmost purity, rigor, and abstraction. BBM’s uncle, Szolem Mandelbrojt, was a founding member of the group; but, later distanced himself from them. Bourbaki held much sway for a while and added a flavor to mathematics as practiced today. [7, 27, 33, 34]

When Paris fell in June 1940, BBM’s father was interned, his Uncle Szolem took asylum in the US, and the remaining family found refuge in central France where Szolem once taught and had a house. There, with the help of excellent tutors who where also displaced, BBM graduate from high school in 1942. Every French student begins their college education in classe préparatoire. These can be considered as cramming colleges (usually adjacent to a major high school) where they prepare for the feared, month-long exams for the French elite universities called Grandes Écoles. However, German occupation tightened and BBM forsook his early college education in favor of survival. For two years, BBM became an itinerant laborer and was self-taught. He worked as a horse groom and an apprentice tool maker while evading the Germans. [4, 33, 34]

BBM found admission to the Lycée du Parc, in Lyons, where he hid in plain sight as a student. In Lyons, at age 19, he realized that when presented with an algebraic or integral (in which he had little training) equation to solve, he could convert it instantly into a geometric shape, manipulate that shape in his mind (intuitively with limited aid from drawings or algebraic calculations), and read the answer. [4, 5, 33, 34]

With Paris’ liberation in August 1944, he went there to continue preparation for the exams. BBM crammed hard but only took the exams for practice. However, he passed—sufficiently in some subjects; but, in mathematics, with the highest score in all of France. He almost ranked first in the competitions for the two leading science schools—École Normale Supérieure and École Polytechnique. [5, 33, 34]

Initially, BBM chose the small and prestigious École Normale—the preferred school for those seeking academic careers. But, the anti-intuitive Bourbaki (with a small office at Normale to this day) was then poised to take over French mathematics. This dogma with its strong biases against geometry and every science was anathema to BBM, who says, he has always had a problem with authority. He knew that his mathematical gift would be stifled by Bourbaki. Normale was the wrong place for a strong-willed person of his tastes. After two days at Normale—he left. [4, 5, 7, 33, 34]
asked to tackle the problem of line noise which could corrupt data transmission. Meanwhile engineers sought to solve the problem by increasing signal strength to drown out the noise. But BBM eventually showed that the noise was both consistent and erratic, some kind of inescapable natural feature of the system that did not disappear with increased signal strength. But more remarkably he also showed that every burst of noise also contained within it bursts of clear signal (a situation he conceived of in terms of the Cantor set). Stranger still, he found that the ratio of periods of noise to periods of clean transmission remained constant, regardless of the scale of time used to plot the phenomenon (i.e., hours, minutes, seconds). In his 1982 book, BBM conjectures that a cross section of Saturn’s rings can be modeled a Cantor set. [1, 4, 14]

**B.3.3 Peano’s curve**

The world thought that Cantor’s mapping of curves onto surfaces was impossible until Peano graphically constructed such a mapping. [14]

Any space-filling curve is often called a Peano curve; see Figure 6c. The one shown here may be the easiest to describe and was developed by German mathematician David Hilbert (heel-bairt, 1862-1943) a year after Peano first invented a space-filling curve. Two examples (third and sixth generations) of the curve are shown for various grid sizes. In each case the curve passes through the midpoint of each grid square. As the grid size goes to zero the num-
ber of midpoints goes to infinity—an uncountable set of points occupying the surface of the square. The curve twists so much in visiting each point on the square that it has infinite length. Thus, there exists a continuous, one-to-one mapping from the points on the curve to those on the plane. In other words, an object with topological (Euclidean) dimension of one can be transformed into an object with topological dimension two through a procedure that should not allow for such an occurrence—simple bending and stretching should leave the topological dimension unchanged. The Peano and Koch (next subsection) curves raised questions about the meaning of dimension. With the discovery of the Peano curves, the question of whether or not there was a mapping function between 1- and 2-dimensional spaces (a homeomorphism). Brouwer proved that this is impossible based on some subtle mathematics that finally established the rigorous notion of topological dimension. [2, 13, 28, 35]

As a physical example of a volume-filling surface, the surface area of the lungs is topologically 2-D and is approximately equal to the area of a tennis court. The fractal dimension of the lungs, however, approaches the volume-filling value of 3. The existence of such disorganized structures in nature often results from functional optimization. Indeed, this is how trees maximize their surface areas. [2, 20]

**B.3.4 Koch’s curve**

Swedish mathematician Helge von Koch’s (hel-guh fon-kohk) curve was published in 1904. Koch was dissatisfied with Weierstrass’ very abstract and analytic definition of a nondifferentiable function. So he gave a more geometric definition of a similar function. The title of his paper was: *On a continuous curve without any tangent, obtained through an elementary geometrical construction*. The two steps of this construction (and many constructions presented here) have been given the names *initiator* and *generator*. In Figure
6d, the initiator is the straight line on the interval [0, 1] corresponding to Step 0. The generator is a template for building the curve (fractal\textsuperscript{42}) and replaces the initiator and, in this case, is the shape shown in Step 1. Now each of the four straight lines in Step 1 become initiators and upon applying a scaled-down version of the generator to each of the four lines the form of Step 2 is obtained. Repeating this process an infinite number of times causes the length of the triadic Koch curve to become infinite—replacing a length with a length 4/3 larger an infinite number of times—while the area \textit{under the curve} approaches some bounded value. This constructed curve is continuous but with no tangent anywhere—just like Weierstrass’ function. [36]

Obviously, the Koch curve is polygonal in form. Most readers have seen curves approximated by a polygon, where, as the number of sides of the polygon increases and as the lengths of the sides decrease, the length of the polygon approaches some finite limit—\textit{e.g.}, the circumference of the circle. Curves having a finite length are termed \textit{rectifiable}. The Koch curve is nonrectifiable. [1]

\textbf{B.3.6 Hausdorff’s fractional dimension}

Felix Hausdorff (German, 1868-1942) is the next mathematician to contribute to the unfolding story of measure and dimension. In 1919, he introduced the notion of \textit{fractional} (as to integer) \textit{dimension} in a seminal paper. Many of the technical developments used to compute Hausdorff’s dimension for highly irregular sets were obtained in 1935 by Abram Besicovitch (1891-1970) who was Russian-born and -educated and then immigrated to England in 1925. Thus, fractional dimension is often called the \textit{Hausdorff-Besicovitch dimension} or simply \textit{Hausdorff dimension}. It is also less frequently called \textit{capacity dimension}. [1, 7]

\textsuperscript{42} Deterministic fractals come into existence after an infinite number of initiator/ generator operations. The stages before reaching this limit are \textit{protofractals}. [9]
Hausdorff’s paper was entitled *Dimension and Outer Measure (Dimension und äußeres Maß)*. In it, he adapts Carathéodory’s “p-dimensional measure for a set in q-dimensional space” so that it makes sense for noninteger p. Hausdorff realized that Carathéodory’s construction for integer dimensions made sense, and was useful, for defining fractional (noninteger) dimensions. He claimed that this is a “small contribution”; however, the abstract formulation of Hausdorff space\(^{43}\) is one of the most important developments of 20\(^{th}\) century mathematics. [11, 13, 24]

After justifying Carathéodory’s measure theory which set forth the conditions for an outer measure, Hausdorff states that it is tempting to extend it to positive, noninteger values of dimension, \(p\). He questions if sets with noninteger dimension would be trivial sets having outer measures of either 0 or \(\infty\)—\(i.e.,\) meaningless values obtained by measuring with an inappropriate ruler. [24]

So what is an inappropriate ruler? One will remember that Peano’s curve was infinitely long when it covered a square. That is using a 1-dimensional ruler to measure a 2-dimensional surface yields \(\infty\). Also using a 1-dimensional ruler, Lebesgue found that the measure of the zero-dimensional end points of a segment had measure 0. An appropriate ruler for a surface needs to be 2-dimensional while such for a point needs to be 0-dimensional. Using an appropriate ruler, the measure of an object is *finite*.

So, considering objects of integer dimension, the Hausdorff \(p\)-dimensional measure (Hausdorff \(p\)-measure), \(H^p\), of a surface \((p = 2)\) is defined by example as follows. Let the surface be a simple rectangle of length \(x\) and width \(y\) yielding an area measure of \(xy\).

---

\(^{43}\) **Hausdorff space**: is a topological space in which points can be separated by neighborhoods. The Hausdorff dimension seems to have undertones of *dimension of a Hausdorff space*, thus suggesting it is a topological concept—which emphatically is not the case. The Hausdorff dimension is a metric dimension and is defined for all metric spaces. It is defined roughly by the statement that if you try to cover your object with balls of diameter \(1/N\) (rulers), you’ll need about \(N^p\) of them. Rulers cause one to cross from topology to geometry (metric space). [1, 7, 14, 36a]
nominal number of discs of radius $r$ (2-dimensional rulers) as defined by Equation 1 to form a cover of the rectangle is $\frac{xy}{\pi r^2}$. The Hausdorff 2-measure (area) of the rectangle is then the number of rulers needed multiplied by the area of each ruler; i.e.,

$$H^2 = \lim_{r \to 0} \frac{xy}{\pi r^2} \times \pi r^2 = xy$$

Now, to examine inappropriate rulers. The Hausdorff 1-measure of the same rectangle using a 1-D ruler is

$$H^1 = \lim_{r \to 0} \frac{xy}{\pi r^2} \times \pi r = \infty ;$$

i.e., an infinite line covers a surface. The Hausdorff 3-measure is

$$H^3 = \lim_{r \to 0} \frac{xy}{\pi r^2} \times \pi r^3 = 0 ;$$

i.e., a surface has no volume. Thus far, Hausdorff measure behaves as one would expect it to. [18, 24, 31]

To measure a 2-dimensional object with a 1.99-dimensional ruler also leads to an infinite measure as using a 2.01-dimensional ruler leads to zero measure. This is the basis of Hausdorff’s measure. He continues in his paper: The problem is then to construct sets for which their measure is finite for a noninteger dimension. … The proof that there exists sets that have exactly a given dimension (i.e., have a corresponding measure which is neither zero nor infinite), is not, however, as natural as our concept of dimension. … Thus the problem arises to construct sets $A$ for which $0 < L(A) < \infty$ and are exactly of dimension $p$. [24]

So, for a noninteger (actually, any real) number $s$, the Hausdorff measure, $H^s$ of set $A$ is nonzero only for the unique dimension $p$ and is otherwise 0 or $\infty$, i.e.,
and \( p = D_H \) is Hausdorff’s fractional dimension. As an example, Hausdorff computes the fractional dimension of the Cantor set of Section B.3.2.

It was, and still is, desired to avoid ugly mathematical developments in this document. However, the foreshortened version of parts of Hausdorff’s paper that follows is intended to illustrate that the calculation of Hausdorff’s dimension is complicated even for simple constructions—and possibly impossible for others. In his paper, he determines the fractional dimension for the entire family of Cantor sets and in particular he reports this dimension for the Cantor middle-1/3 set of Figure 6b. The reader may wish to skip this. [7, 24]

Consider a sequence of positive numbers \( \xi_0, \xi_1, \xi_2, \ldots \) on the closed interval \([0, \xi_0]\) such that, \( \xi_n > 2\xi_{n+1} \) (refer to Figure 7). From the closed interval, delete the central open interval \( i.e., \) do not consider interval end points) of length \( \xi_0 - 2^{1/2} \) and denote it by \( \beta(1/2) \). From the remaining two intervals of length \( \xi_1 \), delete the central open intervals of length \( \xi_1 - 2\xi_2 \) and denote these deleted intervals as \( \beta(1/4) \) and \( \beta(3/4) \) on the left and right respectively as shown in the figure. Thus, for Step \( n \), the open intervals are denoted by \( \beta(y), y = (1/2^n, 2/2^n, 3/2^n, \ldots, 2^{n-1}/2^n) \). [24]
After deleting the set of open intervals \( B = \sum \beta(y) \) from the original closed interval \([0, \xi_0]\), there remains a perfect, nowhere dense set \( A \), which can be covered by \( 2^n \) intervals of length \( \xi_n \) for \( n = 0, 1, 2, \ldots \). The measure (length) of each of these intervals is \( \lambda(\xi_n) \) such that the total measure of the set, \( L(A) \), is [24]

\[
L(A) \leq \lim_{n \to \infty} [2^n \lambda(\xi_n)] \quad \text{as} \quad \xi_n \to 0. \tag{3}
\]

This last statement of outer measure can be better understood by considering the total measure of the continuous curve of Figure 8. Here a 1-D curve is covered by discs of diameter \( \varepsilon \) to form a polygonal approximation to the outer measure of the curve that is greater than or equal to the true measure. As the diameters of the discs go to zero, the true outer measure of the curve is obtained. The infimum (fn 28) of this sum is taken so as to get in some sense the best possible cover with sets of diameter \( \varepsilon \) or less. [18]

Next Hausdorff scales the \( \xi_n \) such that

\[
2^n \lambda(\xi_n) = 1 \tag{4}
\]

Figure 8—A rectifiable curve covered by discs of diameter \( \varepsilon \). Overlapping discs are centered on the curve and segments are drawn through the center of the discs tangent to the curve; thus, the sum of the segment lengths is greater than the length of the line—an outer measure. [after 18]
where $\lambda(\xi_n)$ is the length function of unknown dimension for the individual $\xi_n$ just as $L(A)$ is
the total length function for the set. Thus, combining Equations 3 and 4 he gets [24]

$$L(A) \leq 1$$  \hfill (5)

Next, Hausdorff sets out to show that

$$L(A) \geq 1$$  \hfill (6)

such that if both Equations 5 and 6 are true, then it will follow that \( L = 1 = 2^n \lambda(\xi_n) \). To prove
Equation 6, Hausdorff considers the open intervals $\beta(y)$ to have end points $u(y)$ on the left
and $v(y)$ on the right. The end points are not part of the open interval. Add to these points
$v(0) = 0$ and $u(1) = \xi_0$. Then the length of any $\xi_n$ is simply (refer to Figure 9) [24]

$$u\left(\frac{k}{2^n}\right) - v\left(\frac{(k-1)}{2^n}\right) = \xi_n$$ \hfill (7)

and, because of the $y$-numbering convention of the $\beta(y)$, define $y_3 = (y_1 + y_2)/2 = (2k-1)/2^{n+1}$
and thus Equation 7 can be written as

$$u\left(\frac{k}{2^n}\right) - v\left(\frac{(2k-1)}{2^{n+1}}\right) = u\left(\frac{(2k-1)}{2^{n+1}}\right) - v\left(\frac{(k-1)}{2^n}\right) = \xi_{n+1}.$$  \hfill (8)

Using Equations 7 and 8 as guides and substituting $\eta$ for $y_2$ and $\zeta$ for $y_3$, Hausdorff
claims that the generalized length of any $\xi_{k+1}$ is $\alpha$ which is defined as

$$\alpha = u\left(\frac{k}{2^n}\right) - v\left(\frac{(2k-1)}{2^{n+1}}\right)$$  \hfill (9)

Further, referring to Equation 4 and since

Figure 9—This may be helpful in understanding Hausdorff’s procedure
\[ \eta - \zeta = \frac{k}{2^n} - \frac{2k-1}{2^{n+1}} = \frac{1}{2^{n+1}}. \]

Hausdorff states that Equation 8 satisfies the inequality

\[ \lambda(\alpha) \geq \eta - \zeta. \] (10)

This he proves recursively starting (refer to Figure 9) with \( \zeta = 0, \eta = 1, \alpha = \xi_0, \) and \( \lambda(\alpha) = 1. \) [24]

Finally, he covers the Cantor set with open intervals of length \( \alpha_n < \rho \) such that

\[ \sum \lambda(\alpha_n) < L_\rho + \varepsilon, \]

where \( L_\rho \) is the lower bound of \( \sum \lambda(\alpha_n) \) and \( \varepsilon \) is very small and from Equation 10

\[ \frac{\lambda(\alpha_0)}{y_1} \geq 1 \]
\[ \frac{\lambda(\alpha_1)}{y_2 - y_1} \geq 1 \]
\[ \vdots \]
\[ \frac{\lambda(\alpha_{n-1})}{y_n - y_{n-1}} \geq 1 \]
\[ \frac{\lambda(\alpha_n)}{1 - y_n} \geq 1 \]
\[ \sum_i \lambda(\alpha_i) \geq 1 \]

Thus, \( L_\rho + \varepsilon \geq \sum \lambda(\alpha_i) \geq 1 \) and it follows as \( \varepsilon \to 0 \) that \( L_\rho \geq 1 \geq L \geq L_\rho \) yielding

\[ L_\rho = L = 1 \]

and the set has dimension \([\lambda(x)]\). This completes an abbreviated version of Hausdorff’s proof. [24]

Hausdorff then computes the dimension of the middle-1/3 Cantor set by stating that for a set of dimension \( p \) (\( 0 < p < 1 \)), \( \lambda(x) = x^p \). Then from Equation 4

\[ 2^n \lambda(\xi_n) = 2^n (\xi_n)^p = 1. \] (11)
The Cantor set construction sequence yields \( \xi_n = \left( \frac{\xi_1}{\xi_0} \right)^n \xi_0 \) and for \( \xi_0 = 1 \) the length of an individual element is \( \lambda(x) = (\xi_1)^n \) and Equation 11 becomes

\[
2(\xi_1)^p = 1 \quad \text{or} \quad p = \log \frac{2}{\log(1/\xi_1)} = D_H
\]  

(12)

In Figure 6b, \( \xi_1 = 1/3 \) and the Hausdorff dimension of the set is \( D_H = 0.63903 \). Yet, the topological dimension of an individual point or a countable set of points is 0. But, the Cantor set has an uncountable number of points. [24]

Initially, BBM defined fractal dimension as Hausdorff’s fractional dimension when it exceeded the topological dimension; but, as has just been demonstrated, fractional dimension is difficult to determine for simple, deterministic forms. It would be impossible to compute for the stochastic forms of nature. Thus, equivalent and less complicated methods of determining Hausdorff’s dimension are needed.

**B.3.7 Self similarity**

A shape is exactly self-similar if it can be split into smaller parts where each part is an exact copy of the original shape except that it is scaled by a linear contraction factor, \( r \), and the intersections between parts are empty or small in the sense of dimension (a point or line width has zero dimension). The roughness of these sets is characterized by the self-similarity dimension, \( D_{\text{sim}} \) (§B.3.8.1). [9]

As the smaller parts are of the same form as the original shape (geometric similarity) they are said to be scale invariant, scaling, invariant under internal similarity, dilatation (magnification) invariant, symmetrical under magnification, or self-similar. Exactly self-similar fractals such as a Cantor dust and a Koch curve are self-similar as a portion of either construct can be scaled up to achieve the original form (by virtue of their recursive
constructions)—see Figure 10. Said another way, if one were given a photograph of either construct without the endpoints being shown, they would not know what portion of the object they were looking at—the full object between 0 and 1 or a small portion magnified 1000x. Self-similar fractals are called *scaling fractals.* [1, 7, 9, 26]

The idea of recursive self-similarity was originally developed by Leibniz who even worked out many of the details. It was experimentally developed by Lewis Fry Richardson (mentioned again further on) who, in 1926, postulated that over a wide range of scales turbulence is decomposable into self-similar eddies (see Figure 11). From his turbulence work, Richardson suggests the term *cascade* as the self-similarity generating mechanism. If each piece of a shape is geometrically similar to the whole, both the piece and the generating cascade are called self-similar. The idea of self-similar *curves* was taken further in 1938 by French mathematician Paul Lévy (la-vee, 1886-1971) who considered a new self-similar curve—the Lévy C curve. Analytical consequences of self similarity in mechanics were drawn by Russian mathematician Andrey Kolmogorov (kah-l-muh-gah-rahf, 1903-1987) in 1941. [1, 7, 24, 37]

---

44 Cantor’s and Koch’s construction procedures are cascades. [1]
45 Paul Lévy claimed no students; however, the professor BBM was most aware of at Polytechnique was Lévy. [34]
BBM says that turbulence is best understood as *wild variability and randomness* as contrasted to the mild variability of every-day stochastic (Gaussian\textsuperscript{46}) processes. Nobody has yet explained turbulence; but fractals and multifractals (not considered here), which BBM developed to study turbulence and immediately applied to finance, do provide a more accurate description. [33, 38]

![Figure 11—Self-similar eddies in a turbulent cascade [39, 40]](image)

Relative to fractal geometry, BBM first used the term self-similar in 1964, in an internal report at IBM, where he was then working. In 1965, he presented a paper to the IEEE entitled: Self-similar error clusters in communications systems—line noise modeled by a Cantor set mentioned earlier. [36, 41]

*Self similarity is a property of fractals.* But, not all self-similar objects are fractals. For example, a straight line is exactly self-similar; but, since its Hausdorff dimension does not exceed its topological dimension, it is not a fractal. [1, 6]

**B.3.8 Other dimensions**

Hausdorff’s dimension is the gold-standard dimension used to define fractal dimension. Yet, as observed above, that dimension is extremely difficult to rigorously determine even for a simple construct such as a Cantor set. There are a number of other dimensions each of which yield a unique (yet, often the same) value as the Hausdorff dimension.

\textsuperscript{46} If a measure has a Gaussian distribution, its mean and variance describe the process completely; for other near-Gaussian distributions, a few low-order moments contain most to the information. For [fractals] on the other hand, the first few moments give little clue to their nature. [56]
Some of these dimensions are easier to calculate, more precise than others, and/or better suited to characterize a physical property. [21]

In seeking a dimension as an approximation to Hausdorff’s dimension, the similarity and box-counting dimensions will be examined here.

**B.3.8.1 (Self-) Similarity dimension**

Of the other dimensions, the simplest to compute is the *self-similarity dimension* (usually referred to simply as the *similarity dimension*) which is less general than Hausdorff’s dimension; but, often more than adequate to *approximate* it. It yields in most cases the same value as the Hausdorff dimension. *It applies only to self-similar forms; i.e.,* to most of the well-known mathematical fractals such as the Cantor set and Koch curve. To determine the fractional [fractal] dimension one only needs to define the self similarity. [1, 31]

To establish a feeling for the similarity dimension, consider first sets that are 1-, 2-, and 3-dimensional. For a one-dimensional straight line of unit length, divide the interval into an integer number of smaller intervals, *b*, that do not overlap. This forms \( N = b \) copies of the original or *pavings* of the unit interval. Each of the *N* pavings is a scaled down version of the initial, *whole*, unit interval defined by a *similarity ratio* (aka *linear scale factor*, *contraction ratio*, *shrinking factor*, *scaling factor*), *r*, expressed by [18]

\[
r(N) = \frac{1}{b} = \frac{1}{N}.
\]

As an example, consider the unit interval of Figure 12a. It is divided into \( b = N = 5 \) pavings. The similarity ratio that would shrink down the original interval to the size of an individual paving would then be \( r = \frac{1}{5} \). [1]
For a two-dimensional rectangle, using a linear scale factor *each side* can be divided into \( b \) sublengths. Then the rectangle could be paved with \( N = b^2 \) parts. Again the similarity ratio can be deduced *from the whole* by

\[
\frac{1}{b} = \frac{1}{N^{1/D}} = \frac{1}{N^{1/2}}.
\]

In Figure 12b, each side of the rectangle is divided into \( b = 2 \) sublengths forming \( N = b^2 = 4 \) tiles. Thus, the similarity ratio is

\[
r = \frac{1}{2} = \frac{1}{4^{1/2}}.
\]

For \( D = 3 \), a cube has a similarity ratio of

\[
r(N) = \frac{1}{b} = \frac{1}{N^{1/D}} = \frac{1}{N^{1/3}}.
\]

In Figure 12c, each side of the cube is divided into \( b = 3 \) sublengths forming \( N = b^3 = 27 \) bricks and the similarity ratio is

\[
r = \frac{1}{3} = \frac{1}{27^{1/3}}.
\]

Common to each of these cases is the identity\(^{47}\)

\[
Nr^D = 1. \tag{13}
\]

or

\[
D = -\frac{\log N}{\log r(N)} = \frac{\log N}{\log(1/r(N))}. \tag{14}
\]

where \( D \) is the everyday, garden-variety Euclidean or topological dimension. However, when applying Equations 13 and 14 to the pathological forms already discussed, a

\(^{47}\) In these, the Hausdorff measure, \( H \), equals 1. \( Nr^D = H \).
fractional dimension emerges which is not equal to the topological dimension but rather is equal to Hausdorff’s fractional dimension when the object considered is exactly self-similar. This fractional dimension is called the self-similarity dimension, $D_{sim}$.[1]

Referring first to the Cantor set of Figure 6a, the original whole interval, [0, 1], is divided into $b = 3$ subintervals and the middle segment is removed leaving $N = 2$ parts. The scale ratio of the parts to the whole is $r = 1/b = 1/3$. Applying Equation 14, the similarity dimension is

$$[D_{sim}]_{Cantor} = \frac{\log N}{\log(1/r(N))} = \frac{\log 2}{\log 3} = \log_3 2 \approx 0.63093...$$

This is the value that Hausdorff computed in his paper discussed earlier. Note how much easier the similarity dimension is to compute than the Hausdorff dimension. [1]

A second Peano-type, space-filling curve developed by Hilbert has an initiator that is the diagonal line between (0, 0) and (1, 1) as shown in Figure 13a. The generator is shown in Figure 13b and Figure 13c shows the next generation. It should be easy to visualize this curve filling the square space after an infinite number of generations. The original segment is divided into segments of similarity ratio $r = 1/3$ and the generator is composed of $N = 9$ parts. Of course, the topological dimension of the curve is 1; but the similarity dimension of this curve is

$$[D_{sim}]_{Hilb} = \frac{\log N}{\log(1/r(N))} = \frac{\log 9}{\log 3} = \frac{2\log 3}{\log 3} = 2;$$

i.e., it is plane-filling. [1]
Next, referring to Figure 6c, the initiator of the Koch curve is the line between 0 and 1. It is divided into segments each having \( r = 1/b = 1/3 \) in length and the middle segment replaced with two segments also of \( r = 1/3 \) of the initiator leaving \( N = 4 \) parts. Applying Equation 14 yields

\[
[D_{\text{sim}}]_{\text{Koch}} = \log N / \log (1/r(N)) = \log 4 / \log 3 = 1.26186 \ldots
\]

Yet, the topological dimension of the original initiator or generator is 1. [1]

Skipping ahead to a fractal not yet discussed, the Sierpiński sieve or triangle of Figure 24 has an initiator which is an equilateral triangle. The generator splits this form into three triangles \( (N = 3) \) each with scale factor \( r = 1/2 \). Thus, its similarity dimension is

\[
[D_{\text{sim}}]_{\text{Sier}} = \log N / \log (1/r(N)) = \log 3 / \log 2 = 1.58496 \ldots
\]

After an infinite number of recursive operations, the triangles become points with topological dimension 0. Yet, this uncountable set of points spread over a planar surface has a dimension greater than 1. [1]

If the segment lengths of, say, the Koch curve were covered with spheres having diameter \( \varepsilon \) equal to the length of the segments. As \( \varepsilon \to 0 \), Equation 14 could be written as

\[
D_{\text{sim}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}
\]

Note that this looks similar to the definition of Hausdorff dimension of Equation 12.
B.3.8.2 Box-counting dimension

Appendix D (Mathcad Image Processing) uses the box-counting dimension to compute fractal dimension from digital, black-and-white photographs of various dispersions of vegetal matter.

The Hausdorff dimension is difficult to measure and the similarity dimension only applies to self-similar objects; thus, a more versatile method of determining the fractal dimension of the random sets occurring in nature is needed. This method is the box-counting or box dimension which goes back to the late 1920s. Although not as easily computed as the similarity dimension, it is still far simpler to compute than Hausdorff’s fractional dimension and thus is one of the most-popularly used fractional dimensions. Box-counting dimension is the vernacular for the Minkowski-Bouligand\(^{48}\) dimension (aka Minkowski dimension). [7, 42]

To calculate this dimension for a fractal, imagine the fractal lying on an evenly-spaced grid, and count how many grid squares or boxes are required to cover the set. The box-counting dimension is calculated by seeing how this number changes as one makes the grid finer. Suppose that \(N(\varepsilon)\) is the number of boxes of side length \(\varepsilon\) required to cover the set. Then the box-counting dimension is defined as\(^{49}\)

\[
D_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},
\]

which correspond to the upper limit and lower limit respectively of Equation 14; i.e., the box-counting dimension is

---

\(^{48}\) Named for French mathematician Georges Louis Bouligand (1889-1979) who presented *Ensembles impropropres et nombre dimensionnel* in 1928 and Lithuanian-born, German mathematician Hermann Minkowski (miŋg-kuhf-skee, 1864-1909). Around 1907, Minkowski realized that Einstein’s special theory of relativity (1905, based on previous work of Lorentz and Poincaré) could be best understood in a four dimensional space. This nice representation certainly helped Einstein’s quest for general relativity. Prior to this, Einstein had been Minkowski’s mathematics student. Einstein’s distain for mathematics earned him the nickname lazy dog from Minkowski. By the time Einstein fled Germany, he had also become a refugee from mathematics. He later said that he could not find, in that garden of many paths, the one to what was fundamental. He turned to the more earthly domain of physics, where the way to the essential was, he thought, clearer. [7, 43]

\(^{49}\) If the limit does not exist, then one must talk about the upper box dimension, 
\[
D_{\text{upper}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},
\]

and the lower box dimension, 
\[
D_{\text{lower}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},
\]

i.e., the box-counting dimension is
As examples, consider the point, line, and square shown in Figure 14 and the accompanying grid whose mesh is successively made smaller. The box (grid) size in the upper figure is of unit length on each side while those of the second and third figures are 1/2 and 1/4 respectively.

In all three cases it only takes one 2-dimensional box to cover the point—regardless of the size of the box. That is,

\[
(D_{\square})_{\text{point}} = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = 0. [31, 44]
\]

For the line of unit length, initially one box of unit area covers the line—this is not too informative. As the grid size is reduced to 1/2, two boxes are needed to cover the line; and, similarly as the box size is reduced to 1/4, four boxes are needed to cover the line. Thus,

\[
(D_{\square})_{\text{line}} = \log(2)/\log(1/(1/2)) = \log(4)/\log(1/(1/4)) = \ldots = 1.
\]

For the square, one of the initial squares completely covers it—again, not informative. However, for a grid box of side 1/2, four boxes are needed to cover the square. And, for a grid size of 1/4, sixteen boxes are needed to cover the square. Thus [31, 44].

\[
(D_{\square})_{\text{square}} = \log(4)/\log(1/(1/2)) = \log(16)/\log(1/(1/4)) = \ldots = 2.
\]

In this work, the smallest practical box (ruler) size is a pixel of the digital photographs being analyzed. Whereas, in both Equation 15 and Equation 12 for the Hausdorff dimension, the limit of the ruler size is zero. In examining a digital photo, if \(\varepsilon \to 0\), then

---

well defined only if the upper and lower box dimensions are equal. The upper box dimension is sometimes called the entropy dimension, Kolmogorov dimension, or upper Minkowski dimension, while the lower box dimension is also called the lower Minkowski dimension. [7, 42]
the realized fractional dimension of the pixel (which is much larger than the infinitesimal ruler) would be 2. [31]

The point, line, and square in Figure 14 have been optimally placed to line up with the grids of various refinements (an optimal arrangement of rulers such that infimum and supremum do not need to be considered). Obviously, by shifting the line or the square on the grid, more boxes will be needed to cover the objects. In dealing with digital photos, such shifting of the photo for an optimum coverage is not convenient. [31]

In ignoring the infimum and limit of the Hausdorff measure (Equation 3), Equation 15 reduces the same relation as used for similarity dimension

\[ H = N r^d. \] (16)

Taking the logarithm of both sides of Equation 16 and rearranging leads to

\[ \log N = d \log \left( \frac{1}{r} \right) + \log H. \] (17)

This is the equation of a straight line of slope \( d \) and intercept \( \log H \), and provides a way of calculating both the fractal dimension and the measure. So, one must measure the digital photograph with varying sized rulers and plot the results with the above relation. [31]

The programmed algorithm used to compute the box dimension (as well as lacunarity which is discussed later) is presented in Appendix D (Mathcad Image Processing) along with examples.

**B.3.8.3 Comparison of dimensions**

All of the dimensions discussed so far are related. Some of them, however, make sense in certain situations, but not at all in others, where alternative definitions are more
helpful. Sometimes they all make sense and are the equal. Sometimes several make sense but do not yield the same value. The details can be confusing even for a research mathematician. [45]

Whenever the similarity dimension exists, so does the Hausdorff dimension. Frequently, $D_H = D_{\text{sim}}$. It is known that the topological, Hausdorff, and similarity dimensions are related by

$$D_T \leq D_H \leq D_{\text{sim}}.$$

There is no known criterion characterizing the sets for which the Hausdorff and similarity dimensions will be equal. [1, 18]

The box dimensions and the Hausdorff dimension are related by the inequality [6, 31]

$$D_H \leq D_B \leq D_D.$$

B.4 Real-world examples

The material above discusses the crisis that extended over the years from 1875 to 1925 which led to the formation of modern mathematics—a fruitful period of turmoil and anarchy in mathematics. These constructs were fantasies, deliberately contrived to point out some logical inconsistencies in mainstream mathematics. Now, the reader’s attention is turned to real-world investigations which without knowledge of the material already presented arrived at similar power-law relationships. [1, 2]

As to power laws which have been mentioned earlier, these are simply equations of the form $y = ax^n$ where $y$ is defined in terms of the constant $a$ and $x$ raised to the power $n$. Of course, dimension is used as a power term in computing areas and volumes.

BBM’s PhD thesis had a pivotal affect on his career, as it introduced him to the work of American linguist and philologist George Kingsley Zipf (zif, 1902-1950), an
independently wealthy scholar who taught at Harvard. Zipf decided in the 1940s that the secret of the world resided in a set of mathematical relationships known as power laws. Zipf claimed that power laws characterize everything interesting in the social sciences, and provide them with an element of unity in contrast to the physical sciences. As an example, these laws might relate the size of an event to how often that event occurs. BBM’s central thesis argument became the significance of power laws and their explanation. This largely determined the course of the rest of his life. [5, 33]

After joining IBM in 1958, BBM worked on various computer applications. A very early work there lead to the study of the distribution of wealth in society. It was known that income distributions follow a power law, with low incomes and high incomes related in a specific way. Furthermore, a power law distribution has a special characteristic. Each part of the distribution reflects the whole. Thus, the pattern of relative income distribution is the same in the top half as the pattern in the top quarter of the distribution, which is the same as in the top tenth of the distribution, and so on. [5]

In 1961, BBM was invited to Harvard by Dutch-born, economics professor Hendrik Houthakker (how-tacker, 1924-2008) to give a talk on income distribution. On entering the office of his host, a drawing on the blackboard (that should probably have already been erased) caught his eye. The drawing had the same shape as one representing the income distribution he was about to discuss in his lecture. Was Houthakker also studying income distributions? No, Houthakker told him, the data showed his own research on cotton price fluctuations over 60 years. [4, 5, 7]

How was this possible—synchronicity, serendipity, happenstance, coincidence, Providence? BBM states: ... that drawing ended up occupying half my life. [5]
BBM returned home to analyze cotton prices. Using Houthakker’s 60 years of records and after gathering even more data, he began to perceive an astonishing pattern—one that hearkened back to his earlier work on line noise. He discovered that cotton prices followed a pattern that was both erratic and regular. That is, although price changes were erratic in terms of normal distribution and no one could predict the exact amount of any particular price change, the changes themselves followed a symmetrical pattern with regards to scaling—the pattern was the same whether the time scale was hourly, daily, or monthly. [4]

He started looking elsewhere for the shape he had seen on the blackboard, and soon he began to find it—in the flow of rivers, turbulence in fluids, earthquake magnitudes, oil field reserves. Each was characterized by a motif repeated with variations, and each could be described by its own power law. [5]

By the late 1960s BBM’s attention was drawn to the work of English Nilologist and hydrologist Harold Edwin Hurst (1880-1978). Hurst worked on the design of the Aswan Dam Project and had spent countless years analyzing records of the rise and fall of the Nile River. He needed to know how large a dam would be needed for many successive rainy years—Gaussian statistics could not model this situation. Hurst had developed an empirical power-law model to mimic the Nile; and, BBM added this power-law model to his tool box. Later still, he applied a similar technique for modeling the rise and fall of stock market prices. But the beauty in BBM’s models was that they could generate graphed data whose visual pattern accurately mimicked the visual patterns created by real phenomenon. In other words, they were representing the situation geometrically—just as the BBM had passed his college entrance exams by translating algebraic and integral problems into geometrical problems,
and solving them intuitively. Thus, he realized that there was something orderly and mathematical behind these patterns. That something was self similarity. BBM’s ability to uncover this truth was greatly aided by his position with IBM, where he had access to the tremendous computing power. [4]

All of these factors coalesced when BBM took it upon himself to address the problem of coastline length, raised previously by British scientist Lewis Fry Richardson (1881-1953), who was an ardent pacifist. After retiring from Paisley College of Technology in 1940 he began a major work related to wars. He gathered data on all deadly quarrels since the end of the Napoleonic Wars. He developed a magnitude scale for such quarrels defined to be the logarithm of the number who were killed. He then analyzed a large number of factors associated with such deadly quarrels looking for relations between them. One such factor was: Is there a relation between the frequency of wars and the length of their common frontiers?[13]

While collecting data, he realized that there was considerable variation in the various published lengths of international borders. For example, that between Spain and Portugal was variously quoted as 987 or 1214km while that between The Netherlands and Belgium as 380 or 449km.50,51 Richardson investigated this on his own by walking a pair of dividers along a map of a frontier so as to count the number of equal sides of a polygon, the corners of which lie on the frontier; i.e., how the measured length of a border changes as the unit of measurement is changed52. Some of his results are shown in Figure 15a. Richardson’s thoughts on this topic were found after his death among irrelevant unpublished drafts and published posthumously

50 There is a French saying, that what is true on one side of the Pyrenées is false on the other.[37]
51 Athenian sailors reporting to their Admiralty mentioned the difficulties they encountered in seeking how long is the coast of Sardinia.[37]
52 Richardson’s method is known as the ruler algorithm. [18]

\[ L(\varepsilon) = F \varepsilon^{1-D}. \]  \hspace{1cm} (18)

where,

- \( \varepsilon \) – measuring scale length (divider span) used to survey a frontier
- \( L \) – calculated length of frontier
- \( D \) – BBM’s work identified \( D \) as the fractal dimension of a coastline. Each coastline has its own unique value of \( D \) to quantify how jagged it is. Richardson called \( 1-D \) a characteristic of a frontier. Note, \( D > 1 \) such that \( 1-D \) is negative.
- \( F \) – constant positive prefactor (= 1)

Thus, only the two constants \( F \) and \( D \) (once determined) were sufficient to determine the length of a frontier or coastline using \( N(\varepsilon) = F \varepsilon^{-D} \) intervals of length \( \varepsilon \); i.e., \( L(\varepsilon) = \varepsilon N(\varepsilon) = F \varepsilon^{1-D} \).

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*To Richardson, \( 1-D \) was just a simple exponent, say \( x \), that helped to order his data. But, when BBM unearthed his work, BBM proposed that the noninteger \( 1-x \) should be interpreted as a fractal dimension.* [1]
As the unit of measure decreases the length of the coastline increases indefinitely. [1, 7, 18, 21, 34, 37, 46, 47]

Note that Richardson also included a circle in his calculations. For a polygonal approximation of a circle, the dimension of the circle rapidly approaches a limit of 1 as the segments get smaller. Consequently, Richardson showed that most coastlines differ significantly from smooth (rectifiable) curves having topological dimension of one. Although Richardson presented a method, topology fails to discriminate between different coastlines. [1]

The reader should find coastlines reminiscent of Koch’s tangentless curve which becomes infinitely long as the polygonal segment lengths get smaller. This observation was an important motivation for BBM to connect fractal geometry with nature. As noted above, starting with his dissertation BBM had repeatedly encountered mathematical and empirical power-law relations while researching various topics. These experiences along with Richardson’s work helped him bring together his thoughts on this subject when, in 1967, he published How long is the coast of Britain? Statistical self-similarity and fractional dimension (Science, 155, 636-638; 1967). ][14, 18, 37]

The Koch curve (aka Koch coastline) with its inlets and headlands can be used as a rough yet strong model of a coastline–its dimension of 1.26186… is on the high side of the dimensions of Figure 15a which range from 1.13 to 1.25. It does not offer a problem in being too irregular; but, it is too systematic. This can be loosen up by

![Figure 16–Adding an element of chance to a deterministic fractal](image-url)
introducing chance or shuffling to the basic Koch curve—as shown in Figure 16. Whether the Koch curve is loosened up or not, the similarity dimension of the curve remains the same. This value of $D$ for the Koch curve is not empirical but a mathematical constant. Thus, the argument for calling $D$ a dimension becomes even more persuasive. Coastline lengths (like Koch’s curve) are large and best considered infinite. Thus, if one wishes to compare different coastlines from the viewpoint of their extent, length is an inadequate concept—i.e., if a coastline is infinitely long and one half of it is also infinitely long, one needs a better way to express the idea that the entire curve must have a measure that is two times greater than each of its halves. That is, just as measuring the length of a square (Peano curve) gives an infinite result, attempting to measure a coastline in the wrong dimension gives a useless answer. The alternate descriptive measure of a coastline is $D$ and it is a function of the particular stretch of a coastline chosen and is almost independent of the value of $\epsilon$. So, the small and large details of coastlines exhibit the same cragginess and are geometrically identical except for scale as depicted in Figure 17. [1, 18]

![Figure 17](image)

**Figure 17**—At any resolution, more inlets and peninsulas are visible that were not visible before. Thus as one looks at finer and finer resolutions, they see more and more lengths to be approximated, and the total estimate of length diverges faster than the scale gets smaller—just as with the Koch curve. [18]

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54 Random shuffling is the basic method used to create realistic-looking landscapes using deterministic fractal. [1]
One should not be surprised that scaling (exactly self-similar) forms are limited to providing first approximations of natural shapes to be tackled; *e.g.*, modeling a coastline with a Koch curve which has a fractal dimension of 1.26186…. One must rather marvel that these first approximations are so strikingly reasonable. A *self-similar* object can be

- *exactly self-similar* (aka *scaling*) where due to recursive construction each portion can be considered a reduced-scale image of the whole; *e.g.*, the Koch curve,
- *statistically self-similar*, meaning that each portion can be considered *almost* a direct copy of a reference form—as was done in roughing up the Koch curve. Real-world objects can be handled as *natural fractals* as a statistically (*approximately*) self-similar object where parts of them show the same measured properties at a large but finite range of scales. Another example is turbulence in both the final stage of viscous decay and the large-eddy scales, and attempts have been made to describe it in terms of fractals.

One can achieve an infinite level of detail with fractals, but with natural fractals it suffices often to cut off the very large and the very small detail. Both cutoffs are either present or suspected. Practical coastline measurements (or other natural phenomena) would not use infinitesimal measuring rods as Koch’s curve does and would stop long before they got to the size of a grain of sand–a lower bound of measurement. Practical, upper bounds of measurement also exist in natural measurements; for instance, a 500-mile measuring rod in Figure 15b would completely span the island of Great Britain (passing John o’ Groats) and would have little meaning in coastline measurement. [1, 4]

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55 Land’s End in Cornwall is the most westerly tip of the English mainland. The most westerly tip of Great Britain is Corrachadh Mor, Scotland. John o’ Groats is traditionally acknowledged as the extreme northern point of the isle of Great Britain. *Land’s End to John o’ Groats* is a phrase indicating the traversal of the whole of the Isle of Britain (like the American phrase *from sea to sea*). [7]
In applications to natural science, one usually takes the point of view that the fractals that occur in nature are well-behaved with respect to the calculation of their dimension [14]

**B.5 Les objets fractal**

BBM speaks of three distinct periods in his career. The first was a gestation period starting with his dissertation in 1952 and lasting until 1964.

The second period lasted from 1964 to 1975 and was the most interesting as from the viewpoint of fractal development; but, was personally a very frustrating period. BBM’s list of publications was substantial with widely ranging topics (noise, turbulence, galaxy clustering, commodity prices, and river discharges). Few realized that there was single-mindedness (power-law relations) in these varying subjects and not some immature wandering. However, he soon found that the mention of fractional dimension turned off referees and audiences alike. He needed a way to convince readers that the Hausdorff dimension was something real and not theoretical. When he stumbled on Richardson’s work and recognized instantly that a study of coastlines might lend itself to a Trojan horse maneuver. Indeed, everyone has knowledge of geography, but no one BBM knew professionally had a vested professional interest in facts and theories concerning coastlines and relief. The maneuver succeeded. Everyone was wonderfully objective and receptive to the seemingly wild idea contained in this paper, and as a result, became more receptive to the use of fractional dimension in fields that really mattered. Even with the success of *How long is the coast of Britain?* in 1967, this second period was punctuated by premature fractal manifestos in 1964, 1970, and 1972. The 1972 work laid out a program for studying fractal patterns. [5, 37, 50]
BBM’s ideas remained controversial (although they stirred little discussion, pro or con) until 1975, when he published another manifesto in French entitled *Les Objets Fractal*. At this time, he coined the word fractal. This publication was the turning point from the second to the third period and acceptance of his ideas. Why did acceptance not occur earlier? BBM guesses that the essential role played by computer graphics in the publication opened the way. [5, 50]

BBM once defined fractals as having fractional dimension. Now, he insists on only giving an empirical definition of fractals, no abstract definition being totally satisfactory. To define fractals as having fractional dimension would exclude space-filling curves and the boundary of the Mandelbrot set (Figure 21) which has dimension 2. A definition that would not be so bad would be to say that fractals are objects whose Hausdorff dimension is greater than their topological (Euclidian) dimension but, according to BBM, this seems to exclude a few objects that are real fractals. In the end, all that remains is the notion of self-similarity, which is a hallmark of fractals. [7, 36]

Nonetheless, the Hausdorff dimension is intimately linked to fractals. The use of Hausdorff’s dimension had remained concentrated in a few areas of pure mathematics until BBM was the first to successfully conceive, develop, and apply it as a new geometric description of nature that finds order in chaotic shapes and processes. One of BBM’s goals is to establish fractal dimension in a central position in empirical science, thereby showing it to be of far broader import than anyone imagined. [1, 2]

Of fractals, BBM states: *Do I claim that everything that is not smooth is fractal? That fractals suffice to solve every problem of science? Not in the least. What I’m asserting very*

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56 The fractal dimension is a measure of the space-filling ability of a fractal set. The *codimension*, which is the difference between the dimension of the embedding space and the fractal dimension, is a measure of the sparseness of the fractal. [56]
strongly is that, when some real thing is found to be unsmooth, the next mathematical model to try is fractal or multifractal. A complicated phenomenon need not be fractal, but finding that a phenomenon is not even fractal is bad news, because so far nobody has invested anywhere near my effort in identifying and creating new techniques valid beyond fractals. [38]

Of modeling with fractals, BBM states: A properly parsimonious model is required to have a deceivingly simple input and its output must not merely rephrase the input. It must be rich in unanticipated structure, open unanticipated new issues, and, of course, fit the facts. [32]

In the main body of this work, 2-D digital photographs of vegetation will be studied. Yet the actual vegetation occupies a three-dimensional volume. Similarly, in turbulence, one is usually forced to study a 3-D phenomenon using 1- or 2-dimensional sections (crosscuts); and, one asks: How are fractional dimensions from such sections related to that of the original
object? This situation is usually handled by the additive law. Roughly speaking, if a set is embedded in 3-dimensional space and intersected by a plane, the dimension of the intersected set is one less than that of the set provided that the result is independent of the orientation of the intersecting plane. In like manner, the dimension of line intersections is two less than that of the set if independent of orientation. If the set is sparse, a large number of intersections will be required to establish the properties of the set. [56]

B.6 Other types of fractals

The casual observer familiar with the word fractal quite possibly will not associate the word fractal with the forms discussed thus far. A few of such forms are presented below for information only. Each of these forms is generated in recursive or iterative manner. There are other types of deterministic fractals than those shown here.

B.6.1 Mandelbrot and Julia sets

First some entwined backgrounds. In 1936, BBM’s family moved to Paris from Warsaw\textsuperscript{57} expecting the worst to happen in Poland. His father’s younger brother, Szolem Mandelbrojt (note spelling, 1899-1983), had moved to Paris around 1920 as a refuge from Polish mathematics then being built up as a excessively- and militantly-abstract field by

\textsuperscript{57} To round out BBM’s biography scattered throughout this appendix, mention should be made of his early life in Poland. Before entering college, his life was shaped, he says, by the disasters of this century which repeatedly disrupted his schooling: \textit{Without ever trying, I did very well at avoiding being overly influenced by schools}.

BBM was born in Warsaw to a Lithuanian (a nuance of importance in Eastern Europe) Jewish family. His mother was a physician and his father although a scholar from a long line of scholars, manufactured and wholesaled clothing. His mother decided not to send him to first and second grade because of the epidemics then prevalent in the schools. Instead, he was tutored by the husband of an aunt. Uncle Loterman, who was unemployed because of the Depression, offered to be his tutor. He was an intellectual who despised rote learning, including even the alphabet and the multiplication table: both mildly trouble BBM to this day. However, his uncle trained his memory by paying attention to miscellaneous facts and his mind in an independent and creative way through extensive reading. Most of his time was spent playing chess, reading maps and learning how to open his eyes to everything around him. He once read an entire set of encyclopedias. Certainly, these experiences did not harm and probably even helped the geometric intuition which has been BBM’s most important intellectual tool. [59]
Waclaw Sierpiński\textsuperscript{58} (suhr-pin-skee, 1882-1969, see Figure 24). Szolem became Professor of Mathematics and Mechanics at the Collège de France, succeeding Jacques Hadamard (had-uh-mahr, 1865-1963). [4, 5, 31, 32]

In 1945 when BBM was an undergraduate at École Polytechnique, his uncle introduced him to Algerian-born, French mathematician Gaston Julia’s (1893-1978) important 1918 paper claiming that it was a masterpiece and a potential source of interesting problems. At that time, Julia was Professor of Differential Geometry at Polytechnique but young BBM only saw him from a distance. BBM reacted rather badly against suggestions posed by his uncle since he felt that his whole taste in mathematics was so different from that of his uncle—the uncle was an analyst who came to Paris because it was the cradle of classical analysis—the nephew called himself geometer. The uncle told the nephew that geometry was essentially dead (except in children’s mathematics) and had to be outgrown to make genuine scholarly contributions. It seems BBM did not like the idea of growing up in this fashion. [4, 5, 31, 32]

Instead BBM chose his own very different course through many different sciences which some characterize as highly individualistic or nomadic. In fact the decision by BBM to make contributions to many different branches of science was a very deliberate one taken at a young age. This maverick’s path, however, brought him back in 1979 to overwhelm Julia’s theory of iteration of functions with new questions and to bring it to fuller and well-deserved glory. Although they often did not agree, no one influenced BBM’s scientific life more than his uncle. [4, 5, 31, 32]

Johannes Kepler (1571-1630, in what is now Germany), who had described planetary motion as ellipses and thus defined the order of something unknown, became a role model for BBM, who says: \textit{I wanted to keep far from organized physics and mathematics and instead}  

\textsuperscript{58} Sierpiński’s works later became rich ground for fractal tools. [51]
find a degree of order in some area—significant or not—where everyone else saw a lawless mess. [5, 13, 33]

As a soldier in World War I, Julia was wounded—he lost his nose and wore a leather strap across his face for the rest of his life. Between painful and unsuccessful operations he continued his investigations while in the hospital. In 1918, at age 25, Julia published his beautiful, masterpiece *Mémoire sur l’itération des fonctions rationnelles* (Treatise on the iteration of rational functions)*59*. The paper received the *Grand Prix de l’Académie des Sciences* [but] then fell into thirty years of neglect and increasing scorn, being overly concrete and specialized. [13, 33]

*Julia sets* are fractals and one can marvel at the fact that Julia was able to determine various properties of these sets without a graphic computer. Julia was interested in the iterative properties of the general expression

\[ z^4 + \left[ \frac{z^3}{(z - 1)} \right] + \left[ \frac{z^2}{(z^3 + 4z^2 + 5)} \right] + c. \]

The Julia set is now associated with those points \( z = x + iy \) on the complex plane for which the series

\[ z_{n+1} = z_n^2 + c \quad (19) \]

does not tend to infinity. The variable \( c \) is a complex constant, one gets a different Julia set for each unique value of \( c \). The initial value \( z_0 \) for the series is a point on the imaginary plane. In the broader sense the exact form of the iterated function may be anything, the general form being \( z_{n+1} = f(z_n) \), interesting sets arise with nonlinear functions \( f(z) \) as shown in Figure 19. [36, 51]

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*59* Independently, French mathematician Pierre Fatou (*fah-too*, 1878-1929) studied this same subject in 1919-1920. In short, Julia sets are the boundaries of Fatou domains. [13, 36]
A computer image of the Julia set is created by starting with each pixel corresponding to a point on the complex plane. The complex coordinates of each pixel then represents the starting point, $z_0$, for the iterative calculations of Equation 19. If the repetitive iterations cause the function to diverge, the point is not part of the Julia set. If it converges, that pixel is colored as shown in Figure 20a. This convergence is not always obvious and it may take a large number of iterations to resolve. An added effect is achieved by coloring the point to indicate how fast it converges or diverges to infinity as shown in Figure 20b. [51]

The Mandelbrot set also uses Equation 19, but instead of plotting those values of $z_0$ that do not diverge for a given value of $c$, it plots values of $c$ for $z_0 = 0$. In Figure 21a, the Mandelbrot set is the dark blue center region. [52]

The Mandelbrot set is generally considered to be a fractal. However, only the boundary of it is technically a fractal with Hausdorff dimension of 2. It is self-similar in the sense that small copies of itself can be found at arbitrarily small scales near any point of the boundary of the Mandelbrot set. [7]
Based on computer graphics, BBM originally conjectured that the set was disconnected. However, computer graphics are unable to detect the thin filaments connecting different parts of the set. A common way around this problem, which also results in more aesthetically pleasing pictures, is to color pixels as described above. Since parameters closer to the Mandelbrot set will take longer to diverge, this method will make the connections between different parts of the Mandelbrot set visible. It has been rigorously shown that the Mandelbrot set is connected. [7]

**B.6.2 Diffusion-limited aggregation (DLA)**

DLA was proposed by Witten and Sander (1981, 1983) to simulate carbon particle aggregates formation in a diesel engine. At the center of a grid of square cells, DLA begins by placing a seed particle that is permanently located. Then from a random location on a large circle centered on the seed particle, another particle is released and allowed to randomly...
diffuse through the grid. If the moving particle wanders too far from the seed, it falls off the edge of the grid and another wandering particle is started at a randomly chosen edge point. When a wandering particle reaches one of the four squares adjacent to the seed, it sticks to form a cluster of two particles, and another moving particle is released. When a moving particle reaches a square adjacent to the cluster, it sticks there. Continuing in this way builds an arbitrarily large object called a diffusion-limited aggregate because the growth of the cluster is governed by the particles’ diffusing across the grid. Figure 22a shows a moderate-size DLA cluster. By attracting particles to a line, a form such as Figure 22b can be formed. Both of these forms remind one of plan- and horizontal-view plant silhouettes. [8, 51]

![Figure 22—Examples of diffusion-limited aggregates.](image)

**B.6.3 Iterated function systems (IFS)**

The Mandelbrot and Julia sets fall under the heading of iterated complex polynomials. The deterministic, exactly self-similar fractals of Cantor’s set, Peano’s and Koch’s curves, and Sierpiński’s forms of Figure 24 fall in the category of iterated function systems (IFS). Above, these forms were developed using iterative geometrical techniques, these techniques can also be written mathematically using iterative mapping functions thus the name iterated function systems. The mappings of a Koch curve would be to a) make 4 copies of the initiator which are reduced by one-third of the original length, b) rotate two of these copies (one by +60° and one by –60°), and c) translate each of the four copies to its appropriate location on the generator. [9, 53, 54]
In 1954, Jörg Wunderlich introduced what are now known as iterated function systems. They were developed in their present form by John Hutchinson in 1981 and BBM in 1982 and have been popularized by the 1988 work of Michael Barnsley whose fractal fern appears in Figure 1a. IFS are the foundation of a substantial image compression industry. [7, 9, 55]

**B.7 Clumping**

The desire of the main body of this work is to have an index which reflects the texture and distribution of vegetal material. Indeed, the box-counting dimension used as a fractal dimension appears to be such an index. Yet, BBM says: *texture is an elusive notion which mathematicians and scientists tend to avoid because they cannot grasp it. Engineers and artists cannot avoid it, but mostly fail to handle it to their satisfaction. ... In fact, much of fractal geometry could pass an as implicit study of texture ... except for one complication.* [1, 57]

**B.7.1 The problem with fractal dimension**

An annoying development stems from the fact that objects with identical fractal dimensions can have greatly different appearances or textures. For example, the two Cantor sets shown in Figure 23 each have fractal dimensions of 0.5—the difference in appearance is in the distribution of the gap sizes. Similarly, in Figure 24, the two Sierpiński carpets each have a fractal dimension of ~1.8957. [1, 57]

![Figure 23 —Two Cantor sets after three iterations each having the same fractal dimension, yet of greatly different appearance—the difference between the two sets is in the distribution the gap sizes. For (a), $N = 2$ and $r = 1/4$ yielding $D = \log(2)/\log(1/(1/4))$; while for (b), $N = 3$ and $r = 1/9$. The difference in appearance or texture can be measured by their lacunarity. [after 57]](image-url)
B.7.2 Lacunarity

In studying the fractal form of the distribution of galaxies, BBM distinguished two aspects of texture—lacunarity\(^60\) and succolarity\(^61\) which can both be controlled through gaps which BBM coined as tremas\(^62\). Here is introduced lacunarity which is an explicit tool for studying texture. Lacunarity means gappiness or hole-iness. A object—whether a fractal or not—is termed lacunar if its gaps or intervals tend to be large and/or if the gap sizes are distributed over a greater range. Thus, the lower image of Figure 23 and the left-hand image of Figure 24 are the more lacunar. [1, 58]

Earlier, power-law expressions of the form \(L(\varepsilon) = F^{-D} \varepsilon\) were presented but only the exponent, \(D\), was considered. To study texture, the random prefactor, \(F\), must be considered also. Several distinct approaches to lacunarity and the prefactor have proved worthy of study. However, one must not expect the resulting alternative measures to be identical functions of one another. Lacunarity is a measure of how data fills space. It complements fractal dimension, which measures how much space is filled. [1]

One is tempted to measure the lacunarity of a Cantor dust by the relative length of the largest gap.

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\(^60\) lacunarity: < Latin *lacuna*, gap, hollow, or pool, (> lagoon) < *lacus*; lake. [1]

\(^61\) succolarity: *percolation* means to *flow through* (< Latin *per*, through + *colare*, to filter, sieve > colander) Mandelbrot coined the word succolarity to mean *to almost flow through* (< Latin *sub*, close to, almost). Both lacunarity and succularity are related to the connectivity of a form. [1]

\(^62\) trema: gap, e.g., the middle-third gap cut from the interval in constructing a Cantor set. (> Greek *τρήμα*, hole. The word’s distant cousin is the Latin *termes* meaning termite.) [1]
whereas for the Sierpiński carpet lacunarity tends to vary inversely with the ratio of \((\text{gap perimeter})/(\text{gap area})^{1/2}\). Another promising measure is deduced from gap size distribution. However, the definition of the prefactor \(F\), and thus lacunarity, is somewhat arbitrary. Fortunately, the details do not matter as long as lacunarity goes \(up\) as any reasonably defined prefactor goes \(down\). In all cases, lacunarity increase is due to the collapse of many gaps into a single bigger one. [1]

Before proceeding, the idea of \textit{translational invariance} needs introducing. An infinitely-long straight line can slide upon itself. After translation, every point on the line \textit{covers} a location where there had previously been a point and the line is said to be translationally invariant. However, the Cantor dust is definitely \textit{not} translationally invariant; \textit{e.g.}, if the dust of the lower line of Figure 6b were translated either left or right by \(1/3\) of the original interval, there would be no overlap whatsoever. If it were translated by \(2/3\) of the interval, only one half of the dust points are in common. [1]

So, now a more precise definition is: the lacunarity of a geometric object, such as a fractal, is a measure of its translational invariance. That is, \textit{at a given scale}, how similar are parts from different regions of a geometric object to each other? Low lacunarity geometric objects are homogeneous and translationally invariant because all gap sizes are the same. In contrast, objects with a wide range of gap sizes are heterogeneous and not translationally invariant; they have high lacunarity. Note that translational invariance is highly scale dependent; objects which are heterogeneous at small scales can be quite homogeneous when examined at larger scales or vice versa. \textit{Lacunarity can thus be considered a scale dependent measure of heterogeneity or texture}. [58]
Note that translational invariance is not the same as self-similarity. The Cantor dusts in Figure 6 were generated by a process that guarantees that they are self-similar; i.e., the units at successively finer scales appear identical to the units at the broader scales. However, because the entire sequence has a wide range of gap sizes, it is not translationally invariant. [58]

**B.7.3 Gliding-box algorithm**

A number of algorithms have been suggested to determine the lacunarity statistic. The *gliding-box* algorithm presented here was originally described by Allain and Cloitre in Reference 58. It is a straightforward algorithm for the calculation of the lacunarity of both deterministic and random fractals which consists of analyzing the mass distribution in the set. [57, 58]

Using an example of Reference 58, lacunarity will be demonstrated by example. Figure 25 represents a 12×12 matrix \((M = 12)\) of gray and white pixels. Each pixel has a 50/50 chance of being either gray or white \((p = 0.5)\). Let the gray pixels be assigned a *mass* of 1 while the white ones a mass of 0. Then create the box shown in the upper left-hand corner of the figure. The box is just large enough to focus on 4 pixels arranged 2×2 \((r = 2)\). In the figure, the box contains 2 gray pixels (occupied sites) and is assigned a *box mass* of \(s = 2\). Then the box is caused to *glide* to the right by one pixel and the box mass again determined—in this case the mass is 1. When the gliding box reaches the end of a row, it returns to the left and moves down one pixel. The total number of box masses of size \(r\) determined will be

\[
N(r) = (M - r + 1)^2.
\]
A frequency distribution, \( n(s, r) \), of the number of boxes of size \( r \) containing \( s \) gray pixels can then be determined. This frequency distribution is then converted into a probability distribution, \( Q(s, r) \), by dividing by \( N(r) \); i.e.

\[
Q(s, r) = \frac{n(s, r)}{N(r)}.
\]

Then the first and second moments of the distribution are

\[
Z^{(1)} = \sum S Q(s, r) \quad Z^{(2)} = \sum S^2 Q(s, r).
\]

Finally, the lacunarity, \( \Lambda(r) \), is defined as

\[
\Lambda(r) = \frac{Z^{(2)}}{\left(Z^{(1)}\right)^2}.
\]  \hspace{1cm} (20)

The statistical moments can also be written as

\[
Z^{(1)} = \bar{s}(r) \quad Z^{(2)} = s_s^2(r) + \left(\bar{s}(r)\right)^2.
\]

where, \( \bar{s}(r) \) is the mean and \( s_s^2(r) \) the variance of the number of sites per box. Thus, lacunarity can be written in more familiar terms as

\[
\Lambda(r) = \left(\frac{s_s^2(r)}{\left(\bar{s}(r)\right)^2}\right) + 1
\]  \hspace{1cm} (21)

Lacunarity is thus a dimensionless representation of the variance. [57, 58]

For the matrix of Figure 25,

Table 2 summarizes the lacunarity calculation which

\[
\begin{array}{|c|c|c|c|c|}
\hline
r & s & n(s, r) & Q(s, r) & s Q(s, r) & s^2 Q(s, r) \\
\hline
0 & 3 & 0.02479 & 0.0 & 0.0 \\
1 & 35 & 0.28926 & 0.28926 & 0.28926 \\
2 & 46 & 0.38016 & 0.76033 & 1.52066 \\
3 & 29 & 0.23967 & 0.71901 & 2.15703 \\
4 & 8 & 0.06612 & 0.26446 & 1.05784 \\
\hline
\end{array}
\]

\( N(r) = 121 \)

\[
\begin{array}{c}
\sum Q(s, r) = 1.0 \quad Z^{(1)} = \frac{\sum s Q(s, r)}{1.0} = 2.03066 \\
Z^{(2)} = \frac{\sum s^2 Q(s, r)}{1.0} = 5.02479 \\
\Lambda(r) = \frac{Z^{(2)}}{\left(Z^{(1)}\right)^2} = 1.21568
\end{array}
\]

\[\text{A complete knowledge of all the different moments is necessary to achieve a proper physical characterization of the set [58]}\]
yields $\Lambda(2) = 1.215$. This calculation is repeated over a range of box sizes ranging from $r = 2$ to, usually, $r = M/2$. Then a log-log plot is prepared of lacunarity, $\Lambda$, vs. box size, $r$, to analyze the fluctuations in the mass distribution function. [57, 58]

From Equation 21, it can be determined that lacunarity is a function of the following:

1. The size of the gliding box. As the box size increases, the average box mass also increases and the probability that the box masses will greatly differ from the average decreases; *i.e.*, the relative variance decreases. The same map will thus have lower lacunarities as the size of the box increases; *e.g.*, $\Lambda(3) = 1.093, \Lambda(4) = 1.037, \Lambda(5) = 1.020$, and $\Lambda(6) = 1.011$. Of course, $\Lambda(1) = 2 = 1/p$. These are plotted in Figure 26. [57]

2. The percent of occupied sites, $p$. As the mean number of occupied sites goes to zero, $s_x^2(r)/\bar{s}^2(r)$ goes to $\infty$. Sparse maps are thus more lacunar than dense maps for same box size, $r$. [57]

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Appendix C—Mathcad Image Processing

C.1 A bit of programming documentation

The Mathcad (Version 11, ©1986–2002 Mathsoft Engineering & Education, Inc.) programming used in this work is developed and used in this appendix. This programming calculates percent cover (see Appendix A) as well as fractal dimension and lacunarity (see Appendix B) from a gray-scale, plan-form photo of the vegetation. Explanation and verification of these programs is included in the material following.

First using a photo editor, a color photo must be converted to a gray-scale bitmap as shown in Figure 1. Then the image is processed as described in the Mathcad object below.

Figure 1—Johannes Vermeer’s
tGirl with a Pearl Earring
6. COMPARE FRACTRAN DIMENSION:

Fractal dimension is determined using the box-counting algorithm. The user will choose the rules sizes and determine the Xa, Xb, then plot a plot (a regression of Xa vs. Xb), to determine the fractal dimension of the shape using equation 17 of Appendix B.

As a preliminary test of this calculation, read an image of known fractal dimension. Here the Sierpinski Gasket (FG) 24 of APRC, screen captured, and turned into a bitmap in photo editor, is a fractal dimension of 1.899...

\[ X = \text{READBMP ('C:CUBIC_3300.fmp')} \]

MAKE GRAY-SCALE IMAGE BW

\[ R = \text{grey}(X) \]
\[ B = \text{col} \]
\[ \text{for COL, C = 1 to 65535, B = (XA - X_B) \times 2^{6}\}
\]
\[ \text{for COL, C = 1 to 65535, B = 255 otherwise} \]

\[ \text{LET'S ACCEPT THAT THE IMAGE BEING PROCESSED WOULD} \]

ANALYZE AS SQUARE WITH AN EQUAL NUMBER OF BOXES AND COVERING.

\[ \text{USER INPUT:} \]

Adjust the values of Xa such that 2 \times Xa > R or C above.

\[ \text{YES} = 1 \]
\[ \text{NO} = 0 \]

\[ \text{COVERED} = \text{YES} \text{ IF } XA \geq 1 \]

\[ \text{COVERED} = \text{NO} \text{ IF } XA < 1 \]

\[ \text{COVERED} = \text{YES} \text{ IF } XM \geq 1 \]

\[ \text{COVERED} = \text{NO} \text{ IF } XM < 1 \]

The fractal dimension of the Sierpinski Gasket is 1.899...

\[ \text{FractDim = 1.899316} \]

\[ \text{W = READBMP ('C:CURVE_33000.fmp')} \]

MAKE GRAY-SCALE IMAGE BW

\[ R = \text{grey}(X) \]
\[ B = \text{col} \]
\[ \text{for COL, C = 1 to 65535, B = (XA - X_B) \times 2^{6}\}
\]
\[ \text{for COL, C = 1 to 65535, B = 255 otherwise} \]

\[ \text{USER INPUT:} \]

Adjust the values of Xa such that 2 \times Xa > R or C above.

\[ \text{YES} = 1 \]
\[ \text{NO} = 0 \]

\[ \text{COVERED} = \text{YES} \text{ IF } XA \geq 1 \]

\[ \text{COVERED} = \text{NO} \text{ IF } XA < 1 \]

\[ \text{COVERED} = \text{YES} \text{ IF } XM \geq 1 \]

\[ \text{COVERED} = \text{NO} \text{ IF } XM < 1 \]

The fractal dimension of the curve is 1.565...

\[ \text{FractDim = 1.565765} \]

\[ \text{FractDim = 1.565765} \]

\[ \text{FractDim = 1.565765} \]

\[ \text{FractDim = 1.565765} \]
4. COMPUTE FRACTAL DIMENSION (D)

FRAC TAL DIMENSION IS DETERMINED USING THE BOX COUNTING ALGORITHM. THE USER WILL CHOOSE THE RULER SIZES AND DETERMINE THE Ns. THEN FROM A PLOT (A REGRESSION) OF N vs. (k) DETERMINE THE FRACTAL DIMENSION USING EQUATION 17 OF APPENDIX B.

AS A PRELIMINARY TEST OF THIS CALCULATION, READ IN AN IMAGE OF KNOWN FRACTAL DIMENSION. HERE THE SIERPINSKI SIEVE (FIG. 24 OF APPX. B) HAS BEEN DRAWN IN MS WORD, SCREEN CAPTURED, AND TURNED INTO A BMP IN A PHOTO EDITOR. ITS A FRACTAL DIMENSION IS 1.58464....

MAKE GRAY-SCALE IMAGE B&W

R = new(M) | R = 1814 | C = color(M) | C = 2004
for | i = 0... R - 1 | j = 0... C - 1

MAKE GRAY-SCALE IMAGE B&W

M = new(M) | M = ROW(C) | C = R
for | COI = 0... C - 1

MROWicol = 0 if MROWicol < 127
MROWicol = 255 otherwise

Since this algorithm is only intended covering the black pixels, we can have as much “white space” around the image as desired. If the number of columns or rows is too high, then it should be easy to repeatedly increase the number of tiles by 2. For this Sierpinski sieve example, we want

1024 rows and columns since R & C = 716.

LET’S ASSUME THAT THE IMAGE BEING PROCESSED WILL ALWAYS BE SQUARE WITH AN EQUAL NUMBER OF ROWS & COLUMNS.

► USER INPUT:

Adjust the value of n such that 2^n is larger than either R or C above: n = 12 2^12 = 4096

Define a “white area”, as Matrix MM

i = 0... | j = 0... | MM[ui, uj] = 255

Read Matrix M into part of Matrix MM leaving white edges.

MM[ui, uj] = | M[i, j] |

Save a copy of this image. WRITEBMP (“C:\SIEVE_NEW.bmp”)...MM

For a square tile (ruler) of a specified size covering a specified portion of the image (specified by the values in the call string of the function), the following function returns a YES if the tile covers all one black pixel of the portion of the image is covered. Otherwise, a NO is returned.

YES = 1 | NO = 0 | COVERED = | NO

for | c = Rstart... Rend

for | k = Cstart... Cend

COVERED = YES if MM[i, j] = 0

COVERED

The following is the routine to determine the number of nonoverlapping tiles to cover all black pixels. This function is called by the user with the information of the ruler size (2^n x 2^n pixels) and returns the number of covering tiles, N, to cover all of the black pixels.

Below is the number of pixels along the side and height of a tile.

NUM_TILES = | NUM = 0

SIZE = 2^n

for | i = 0... 2^n - 1

for | j = 0... 2^n - 1

MM[i, j]

Use only the smallest size tile to determine fractal dimension. The very smallest tile is 2 x 2 matrix elements (pixel).

k = 0.4 | Nk = | NUM_TILES(k - 1) | log(k) = log(Nk) | \( k^{1.5} = \log(\frac{1}{Nk}) \)

m = slope(log10(Nk))

FractDim = m

The actual fractal dimension of the image is 1.5846... Here a fractal dimension of 1.5054 is computed. A line of this slope is drawn in the plot.

► USER INPUT:

Adjust the value of n such that 2^n is larger than either R or C above.

n = 11 2^n = 2048

Define a “white area”, as Matrix MM

i = 0... | j = 0... | MM[ui, uj] = 255

Read Matrix M into part of Matrix MM leaving white edges.

MM[ui, uj] = | M[i, j] |

Save a copy of this image. WRITEBMP (“C:\KOCH_CURVE_300DPI.bmp”)...MM

For a square tile (ruler) of a specified size covering a specified portion of the image (specified by the values in the call string of the function), the following function returns a YES if the tile covers all one black pixel of the portion of the image is covered. Otherwise, a NO is returned.

YES = 1 | NO = 0 | COVERED = | NO

for | c = Rstart... Rend

for | k = Cstart... Cend

COVERED = YES if MM[i, j] = 0

COVERED

The actual fractal dimension of the Koch curve is 1.26186... Here we computed a value of 1.2599318

FractDim = 1.2599318
Lacunarity is computed by the Sliding-Box algorithm of Section 7.3 of Appendix B. To begin to verify this algorithm, use the example of that section that is shown in Figure 25.

The next to consider a 12 x 12 matrix to represent the figure. \( \text{ROI} = 12 \) \( \text{ICOL} = 12 \) \( i = 0, 11 \) \( j = 0, 11 \) \( W = 255 \) (WHITE) \( B = 0 \) (BLACK)

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
0 & B & W & W & W & W & W & W & W & W \\
\end{array}
\]

\[
\text{LAC} = \text{BOx} \times \text{ROI} \times \text{ICOL}
\]

5A. ANOTHER LACUNARITY EXAMPLE

Reference 37 of Appendix B states: “As described by Allan and Clette (1991), the lacunarity curve for self-similar fractals should be a straight line with a slope equal to 0 – 6, where 0 and 6 are the fractal and Euclidean dimensions respectively.” Let’s see how this works for lacunarity here.

\[
\text{MI} = \text{RED} \times \text{BLUE} \times \text{GREEN} \times \text{GRAY}
\]

For \( \log(r) < 0.7 \) \( [r < 5] \), the two curves do not match. This is because some of the pixels of the original grayscale photo of the sieve were indeed gray meaning that the pixel overlapped a boundary between white and black regions. In converting the grayscale image to a black-and-white image the fractal structure of the photo is lost.
C.2 Processed photographic data

Using the above material, the percent cover, fractal dimension, lacunarity, and average size of gaps of the various plant types and arrangements are computed.

C.2.1 Plants 1

Plants 1 were not a uniform dark green and were painted to give good contrast to the back lighting. The paint flaked onto the Plexiglas frame and could have made a difference in the calculations if some masking had not been performed on the gray-scale photos. Values for negative images are shown in the insets—midpage at the right.
In Arrangement 1B, below, a little more masking should have been done to remove the lines at the lower-left hand corner and the left edge.
1. READ GRAY-SCALE IMAGE FROM FILE & CONVERT TO BW

2. COMPUTE PERCENT COVER (PCT)

3. COMPUTE FRACTAL DIMENSION

USER INPUT: Adjust the value of n such that 2^n is larger than either R or C above. n = 11 2^{11} = 2048. Define a "white area" as any pixel with

\[ i = 0 \quad j = 0 \quad j = 0 \]

Using the negative of the above image and [0, 0] = 0

\[ PCT = 0.600 \quad m = 1.934 \quad b = 0.019 \]

\[ \text{Log-log plots: } 250.3 < 1555.2 < 6155.2 \]

\[ \text{Back calc: } 29 \quad 50 \quad 100 \quad \lambda_1 = 1.371 \quad \lambda_2 = 1.294 \quad \lambda_3 = 1.223 \]

\[ \text{The values of } \lambda_j \text{ are: } \lambda_1 = 3.722 \quad \lambda_2 = 6 \quad \lambda_3 = 6 \]

\[ \text{The value of } \lambda_c \text{ is: } \lambda_c = 3.722 \]

\[ \text{The value of } \lambda_0 \text{ is: } \lambda_0 = 0.996 \quad \lambda_0 = 0.975 \]

\[ \text{The value of } \lambda_{\text{max}} \text{ is: } \lambda_{\text{max}} = 1.744 \]

\[ \text{The value of } \lambda_{\text{min}} \text{ is: } \lambda_{\text{min}} = 1.744 \]
1. READ GRAY-SCALE IMAGE FROM FILE & CONVERT TO RAW

M = READIMG("C:\1OG\GRAY\SCALE.IMG")
ROW = round(M)
RCOL = 1.0 + 10^2
ICOL = selcol(M)
ICOL = 1470

2. COMPUTE PERCENT COVER (PCT)

PCT = 0 for ROW = 0, RCOL = 1
PCT = 1 for COL = 0, VICOL = 1
PCT = PCT + 1 if RROW, COL = 0
PCT = PCT for VICOL

3. COMPUTE FRACTAL DIMENSION

\[ FracDim = m \]

4. COMPUTE LACUNARITY

\[ LAG = 100 \times \frac{ROW - ICOL}{ICOL} \]

5. USING THE NEGATIVE OF THE ABOVE IMAGE AND M10 = 0

PCT = 0.669 m = 1.041 b = 6.533

6. SUBSEQ = 205.7 1655.6 60390.5

7. E = 2

\[ M = LAG/\text{SUBSEQ}(ICOL, JCOL) \]

8. 1.0 2.0 3.0

\[ \begin{align*}
& \text{for } \text{ICOL, } JCOL = 1
& \text{else } \text{ICOL, } JCOL = 0
\end{align*} \]

9. FOR E < 1.0, M10 = M

10. FOR E > 1.0, M10 = E

11. FOR E = 1.0, M10 = E

12. FOR E = 0, M10 = E

13. FOR E = LAG, M10 = E

14. FOR E = SUBSEQ, M10 = E

15. FOR E = LAG, M10 = E

16. FOR E = SUBSEQ, M10 = E
1. READ GRAY-SCALE IMAGE FROM FILE & CONVERT TO RAW

2. COMPUTE PERCENT COVER (PCT)

3. COMPUTE FRACTAL DIMENSION

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1. READ GRAY-SCALE IMAGE FROM FILE & CONVERT TO B/W

2. COMPUTE PERCENT COVERED (PCT)

3. COMPUTE FRACTAL DIMENSION

4. COMPUTE LACUNARITY

Using the negative of the above image and $M_{0}=0$

\[
PCT = 0.375 \quad \text{and } M_{0} = 0.666 \quad \text{and } B = 6.428
\]

\[
0.204 > 0.464 > 1.400
\]

\[
S\text{SUM} = 140.7 \quad 870.1 \quad 3206.5
\]

\[
Wg_{(i)} = 0.5 \quad 2.33 \quad 1.92
\]

\[
0.032 \quad 0.124 \quad 0.016
\]
C.2.3 Plants

1. READ GRAY-SCALE IMAGE FROM FILE & CONVERT TO grayscale array
   $$M = READBMP("COLA_GRAYSCALE.bmp", 1)$$
   $$ROW = 1271$$
   $$ICOL = 1412$$

2. COMPUTE PERCENT COVER (PCT)
   $$PCT = \frac{ROW \times COL \times (M \times 100)}{ROW \times COL}$$

3. COMPUTE FRACAL DIMENSION
   $$\text{User Input:}$$ Adjust the value of $n$ such that $2^n$ is larger than either $R$ or $C$ above.
   $$n = 11$$
   $$R = 2^n = 2048$$
   $$C = 2^n = 2048$$

4. COMPUTE LACUNARITY
   $$\text{Lac} = \frac{SUM - (ROW \times COL)}{SUM}$$
   $$SUM = 0$$
   $$SUM = 0$$

5. USING THE NEGATIVE OF THE ABOVE IMAGE AND $M_{2,3} = 0$
   $$PCT = 0.012$$
   $$m = 0.595$$
   $$b = 6.573$$

6. SUMMARY
   $$S = 32.27$$
   $$1699.0$$
   $$7653.2$$

7. USING $M_{2,3} = 0$

8. LOGARITHM
   $$\log_{10}(x)$$

9. LOGARITHM
   $$\log_{10}(y)$$

10. LOGARITHM
    $$\log_{10}(z)$$
C.2.5 Plant_5
C.3 Photographic data processing

C.3.1 Positive images
C.3.2 Negative images—using the percentage cover values of the **positive images**—Lacunarity above is a *mass distribution function* not a *gap distribution function*. Thus, the values of the variables $D_{Av50}$ and $D_{Av100}$ above do not go to zero as PCT goes to 1. So the same calculations were made using the negatives of the images used in Section C.3.1. The calculated values for each negative image is included as an inset on the calculation sheet for each plant arrangement above.
C.4 Discussion of results

1. **Quality of the photographs:** Vegetation of varying types and arrangements was placed in holes of an acrylic frame (Figure 2). The frame was placed in a plywood light box (Figure 3) having fluorescent lights in the base and a mirror set at a 45° angle on top (Figure 4). Then from about 60 feet away a digital color photograph was taken of the back-lit vegetation (Figure 5). A typical back-lit photo is shown in Figure 6. Those factors affecting the quality of photos are:

   a. The intensity of the back lighting was lacking and photos needed to be cleaned up to remove unwanted shadows (see Figure 6). Much brighter back lighting *may* produce photographs with better contrast and the scatter shown in the plots *might* be less.

   b. Camera distances might have varied $\pm 6$ inches over a shooting distance of 60 feet.

   c. The 60 feet camera distance was to minimize the tapering effect of perspective in the photos. A true plan-view photo would have no perspective.

   d. Indicators (say, small red lights) at a known distance apart should be added to the experiment so one can more easily scale the number of pixels per inch.
2. **Quality of the plots:** To this person’s knowledge, percentage cover can presently be determined from aerial and satellite imagery and is a variable commonly used by ecologists. However, fractal dimension, lacunarity, and the average gap diameter computed cannot easily be determined from such imagery. Thus, it would be convenient to be able to predict fractal dimension, lacunarity, *etc.* from percent cover. The plots in the next two subsections demonstrate strong trends when plotted against percentage cover. These trends have been fitted to regressed curves as shown. The plots show that the variables considered can be represented as functions of the percentage cover.

### C.4.1 Positive images

Although the negative images, discussed in the next subsection, are of greater interest, calculations were first made for the positive images and regressed in Section C.3.1. The plots of that section indicate that:

1. **Plot 1—Fractal Dimension vs. Percentage Cover:** For each of the three plants investigated, there seems to be a linear relationship between fractal dimension and percentage cover. These relationships are shown as regressed lines on the plot. Plants 1 and 2 have long tapered grass-like leaves and *might* be combined into one regressed line for...
both. However, Plant 3 had broader leaves and a different regressed curve is needed to describe their data. The plot shows three separate regressed curves (one for each plant type considered) which tend toward a fractal dimension of 2 as the percentage cover goes to 1. One can conclude that the fractal dimension of the plant arrangement can be determined from percentage cover if the plant type is known. The heavier line to the left will be discussed below under Item 2 next.

2. **The heavy line at the left of Plot 1:** The first arrangement investigated for each of the three plants was that of a single plant. The plot of fractal dimension versus percent cover for these three arrangements/plants indicated that there might be a linear relationship between these two variables for a single plant. To investigate this, Arrangements 4A and 5A were studied—see Figure 6. Plot 1 of Figure 6 shows that the suspected relationship does not exist; however, in Plots 2 and 3 the lacunarity values seem to fit the trends of the regressed curves developed in Section C.3.1.

![Figure 6—Investigation into Fractal Dimension for Single Plants](image-url)
3. **Plots 2 and 4—Lacunarity:** The lacunarity computed here is for the vegetal mass and not for the gaps between leaves and plants. Nonetheless, definite relationships are indicated between lacunarity (computed for 50 and 100 pixel box sizes) and percentage cover. Lacunarity is a function of the size of the 2-D ruler used to calculate it. In the example at the bottom of Page 147, it was shown that for a mathematical fractal, the lacunarity curve should have slope $D - E$. However, in that example, the curve does not have that slope for small values of the ruler, $rr$. It seems that any lacunarity value used should be a value where the curve has slope $D - E$. This is the case for Plants 1 and 2 for ruler sizes greater than 50 or 100 pixels; but, not for Plant 3 which only has a point of tangency around 20 pixels. Typical lacunarity plots for these three plants are shown below.

![Figure 6—Typical Lacunarity Curves for Each of the Three Plant Types Considered](image)

4. **Plots 3 and 5—Average Diameter of a Gap:** When determining a friction factor vs. Reynolds number, a characteristic diameter of an equivalent pipe through the vegetation is desired. Lacunarity is a dimensionless value; thus, the Mathcad lacunarity routine used above also returned the value SUMsQ. This is the first moment or mean value of the number of black pixels for a box of a given size. Knowing the number of pixels per inch (PPI), the average diameter of a (assumed round) gap, either $D_{Av50}$ and $D_{Av100}$, were determined. This variable plotted smoothly and regressed well. However, the plot showed the average diameter increased as percentage coverage (PCT) increased. This is
not what is desired. Obviously, the lacunarity being calculated is based on the black mass of pixels and not the white ones of the gaps. Thus, in order to investigate the actual gaps, negative images were investigated. These are discussed in the next subsection. A plot (not shown) of the box-counting measure (Eq. 17, Appendix B) has the same shape as these.

**C.4.2 Negative images**

1. The plots discussed next were made against the percentage cover of the positive images just discussed as this is the value of percentage cover that is used by conservationists and reported by aerial surveys.

2. **Plot 1—Fractal Dimension vs. Percentage Cover:** Again the fractal dimension curves for each plant when plotted against percentage cover regress as straight lines. This time, as negative images are being considered, these lines want to pass through the upper left-hand corner of the plot—as they should.

3. **Plots 2 and 4—Lacunarity:** Again, definite relationships are indicated between lacunarity and percentage cover.

4. **Plots 3 and 5—Average Diameter of a Gap:** Again the data plots smoothly and regresses well. Now, as is desired, the average gap diameter decreases as the percentage cover increases. Here too, a plot (not shown) of the box-counting measure (Eq. 17, Appx. B) has the same shape as these plots.

**C.5 Another look at the box-counting algorithm**

Here is investigated a black square of fractal dimension 2 on a white field. This does not get computed well—possibly due to the extent of the white field. The reader will recall that for a given positive photograph, a white field was added such that the size of the
photo to be processed was some power of 2 on each side. Below a photo was created that is exactly $2^{10} = 1024$ pixels on a side. Using a photo processor, black rectangles of increasing size were drawn in the center of the photograph. The box-counting dimension vs. percent cover for these various photographs is plotted below.

At this time, this phenomenon will not be further investigated.
Appendix D—MATHCAD FRICTION-FACTOR CALCULATIONS

This appendix presents the calculations of friction factor vs. Reynolds number for each plant arrangement and attempts to establish a relationship between friction factor curves based on fractal dimension, lacunarity, and average gap size. Mathcad (Version 11, ©1986-2002 Mathsoft Engineering & Education, Inc.) is used for the programming.

D.1 Establish calibration curve—Here are found the consistent calibration curve data
### Calibration Data

<table>
<thead>
<tr>
<th>m</th>
<th>V0</th>
<th>V1</th>
<th>V2</th>
<th>V3</th>
</tr>
</thead>
<tbody>
<tr>
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</table>

Mean Voltage Values:

\[ V_{\text{mean}} = \frac{1}{n} \sum V_i \]

Standard Deviation for \( V_i \) (Small Samples):

\[ \sigma = \sqrt{\frac{1}{n-1} \sum (V_i - V_{\text{mean}})^2} \]

5% Sensitivity Region:

\[ V_{\text{mean}} \pm 0.002 \]

47% Sensitivity to a Percentage of the Average Voltage:

\[ V_{\text{mean}} \pm 0.01 \]

Plotting all 5 of these calibration curves on one graph shows consistency as shown below. The last calibration curve will be used simply because it was constructed over a wider range of calibration weights. To get a drop force from a voltage, define \( V_{y} \) as:

\[ V_{y} = V_{\text{out}} - V_{\text{in}} = \text{arccos}(V_{\text{in}}/V_{\text{out}}) \]
D.2 Basic calculations for each plant arrangement—This section contains basis drag force and wind velocity data as well as the calculation of Reynolds number and effective friction factor for each plant arrangement.
### ARRANGEMENT 1E

**Data**
- From the Negative Photographic Data of Ascension: $C_{PCT} = 3.12$
- Furniture at Top of Unit (Visible): $W_{F} = 1.87$
- $\lambda_{CT} = 1.03$
- $\lambda_{CT} = 1.00$
- $\Sigma = 1.975$
- $C_{PCT} = 2.58$
- $C_{PCT} = 2.457$

**Loadings**
- No. of Loadings: 3
- $W_{F} = 1.38$
- $W_{F} = 1.06$
- $W_{F} = 0.96$
- $W_{F} = 0.88$
- $W_{F} = 0.76$
- $W_{F} = 0.68$
- $W_{F} = 0.58$
- $W_{F} = 0.48$

**Wind Tunnel**
- Percentage of Max. RPM: $20\%$

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</tbody>
</table>

**Arrangement in**

**Plot**

**Diagram**

---

**Comments**

Each cluster of computed points is generated for the particular plant arrangement. The computed points are generated by operating the wind tunnel at some percentage of its maximum rpm. There is scatter in these clusters of both $X$ and $Y$, so use the average value of $X$ and $Y$ and square fitting to determine values of $a$ and $b$ for each average value of $X$.
### ARRANGEMENT 1M

#### Date
From the negative photographic data of Appendix C: \( \Delta P_{CT} \times 0.472 \)

#### Fundamentals at Top of Massed Plants

- \( b_1 = \text{Influence} \times \Delta P_{CT} \times 0.472 \)
- \( \Delta P_{CT} = 0.57 \times (b_1) \)
- \( b_1 = 1.91 \)
- \( \Delta P_{CT} = 1.75 \text{h} \)

#### No. of Meanings
- \( b_2 = \text{Landing} \times 0.1 \times 1 \)
- \( \text{Landing} = 0.05 \times 0.1 \)
- \( b_2 = 0.102 \)

#### Wrist Tunnel

- \( V_{W,Tunnel} = V_{TCD_{P1}} = \text{Position}_Y = \text{Position}_X = 0.1 \times 1 \)
- \( V_{TCD_{P1}} = 0.01 \times 1 \)

#### TABLE 2.4

<table>
<thead>
<tr>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( d_3 )</th>
<th>( d_4 )</th>
<th>( e_5 )</th>
<th>( e_6 )</th>
<th>( f_7 )</th>
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<td>1.91</td>
<td>0.102</td>
<td>0.01</td>
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<td>0.01</td>
<td>0.01</td>
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</tr>
</tbody>
</table>

#### Validity Through Plants

- \( V_{W,Tunnel} = V_{TCD_{P1}} = \text{Position}_Y = \text{Position}_X = \) (valid)

**EACH CLUSTER OF COUPLED POINTS IS GENERATED FOR THE PARTICULAR PLANT ARRANGEMENT AND IS GENERATED BY OPERATING THE WIND TUNNEL AT SOME PERCENTAGE OF ITS MAXIMUM RPM. THERE IS SOFT TREATMENT IN THESE CLUSTERS OF BOTH \( k \) AND \( \Delta P_{CT} \). USE THE AVERAGE VALUE OF \( k \) AND SPICE FITTING TO DETERMINE VALUES OF \( \Delta P_{CT} \) FOR EACH AVERAGE VALUE OF \( k \).

**Standard Deviation for \( k_{ij} = \sum \text{Small} \text{Solutions} \)

- \( \text{Small} \text{Solutions} = \text{Intercept} \times \text{Intercept} \times \text{Intercept} \times \text{Intercept} \times \text{Intercept} \times \text{Intercept} \times \) (valid)

**CREATE 20 LOAD POINTS SO HAVE 5 LOADS**
### ARRANGEMENT 1Q

**Date:** From the Negative Photographic Data of Appendix C: xct = 0.007

**Fundamentals:**
- Flow rate = η<br> - Voidage = α<br> - L/d = l/d<br> - U/D = U/D<br> - d = d<br> - E = E

**Variables:**
- n = number of repeats
- k = number of trials
- L = length of plant
- d = diameter of plant
- U = velocity of plant
- D = diameter of air
- E = efficiency of plant

**Wind Tunnel Constants:**
- Tc = temperature constant
- Vb = velocity constant
- R = resistance constant

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Tc</td>
<td>2.55</td>
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<tr>
<td>Vb</td>
<td>3.12</td>
</tr>
<tr>
<td>R</td>
<td>3.00</td>
</tr>
</tbody>
</table>

**Equations:**
- \( Vb = \frac{\eta U}{d} \)
- \( Tc = \frac{\eta U}{d} \)
- \( R = \frac{\eta U}{d} \)

**Experimental Data:**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Parameter 1</th>
<th>Parameter 2</th>
<th>Parameter 3</th>
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<td>3.00</td>
</tr>
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<tr>
<td>4</td>
<td>2.55</td>
<td>3.12</td>
<td>3.00</td>
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</tbody>
</table>

**Analysis:**
- Each cluster of data points is analyzed separately for the particular plant arrangement. The average value of each parameter is calculated.
- The standard deviation is calculated using the formula:
  \( \sigma = \sqrt{\frac{1}{n-1} \sum (x_i - \bar{x})^2} \)

**Graphical Representation:**
- A graph showing the relationship between parameters.

---

**Values as a Percentage of the Mean Value:**

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
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<td>40.10%</td>
</tr>
<tr>
<td>2.39</td>
<td>30.10%</td>
</tr>
<tr>
<td>2.23</td>
<td>20.10%</td>
</tr>
<tr>
<td>2.04</td>
<td>10.10%</td>
</tr>
</tbody>
</table>

---

**Figure:**
- A diagram illustrating the plant arrangement and its parameters.
### Arrangement 2C

**Data**
- From the negative photographic data of Arrange C: \( PCT = 0.047 \)
- Functions at top of floral bristles: \( \theta_1 = \text{Brent} \), \( \text{PCT} = 0.047 \)
- \( \text{PCT} = 0.047 \) at 108(PCT)
- \( \text{Dau} = 108 \) at 109(PCT)
- \( \theta_1 = 1.98 \) at 109(PCT)
- \( \text{Dau} = 109 \) at 2.33(PCT)

#### Arrangement 2C
- No. of Students: 12
- No. of Observations: 3
- No. of Experiment: 3
- No. of Randomization: 3
- No. of Observations: 3

#### Plant Arrangement
- Plant Silhouette

### Vectors Through Vectors Through

\[ V_{\text{through}} = \frac{V_{\text{through}}}{V_{\text{through}}^2} \]

### Effective Position Factor

\[ V_{\text{effective}} = \frac{T_{\text{effective}}}{V_{\text{through}}^2} \]

### Error Values

\[ T_{\text{error}} = \text{error}(\text{vector} \cdot \text{vector}) \]

### Each Cluster of Computed Points is Generated for the Particular Plant Arrangement

- \( \theta_1 = \text{Brent} \), \( \text{PCT} = 0.047 \)
- \( \theta_1 = \text{Brent} \), \( \text{PCT} = 0.047 \)
- \( \theta_1 = \text{Brent} \), \( \text{PCT} = 0.047 \)
- \( \theta_1 = \text{Brent} \), \( \text{PCT} = 0.047 \)

### Graphs

- Graph 1: Scatter plot with regression line
- Graph 2: Line graph with data points

### Values

- \( T_{\text{effective}} = \text{time}(\text{vector} \cdot \text{vector}) \)
- \( V_{\text{through}} = \text{velocity}(\text{vector} \cdot \text{vector}) \)
- \( V_{\text{effective}} = \text{effective}(\text{vector} \cdot \text{vector}) \)

### Table

<table>
<thead>
<tr>
<th>( T_{\text{effective}} )</th>
<th>( V_{\text{through}} )</th>
<th>( V_{\text{effective}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2.33 \times 10^{-4} )</td>
<td>( 2.33 \times 10^{-4} )</td>
<td>( 2.33 \times 10^{-4} )</td>
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<tr>
<td>( 2.33 \times 10^{-4} )</td>
<td>( 2.33 \times 10^{-4} )</td>
<td>( 2.33 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

### Additional Observations

- \( \theta_1 = \text{Brent} \)
- \( \text{PCT} = 0.047 \)
- \( \theta_1 = \text{Brent} \)
- \( \text{PCT} = 0.047 \)

### Conclusion

- Each cluster of computed points is generated for the particular plant arrangement by applying the vector through some percentage of the maximum value. There is scatter in these clusters of both \( \theta_1 \) and \( \text{PCT} \). Use the average value of \( \theta_1 \) and \( \text{PCT} \) to determine values of \( T_{\text{effective}} \) for each average value of \( \theta_1 \).
### Arrangement 2E

**Data**

From the negative photographic data of Appendix C: $\phi_T = 0.261$

Functions at Top of Plant Arrangement:

- $\phi_T = 0.261$
- $\phi_T = 0.139$

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<th>No. of Test Plants</th>
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</table>

Each cluster of computed points is generated for the particular plant arrangement and is obtained by operating the wind tunnel. At some percentage of its maximum span, there is scatter in these clusters of both $V_{leading}$ and $\phi_T$. Use the average value of $V_{leading}$ and $\phi_T$ for each average value of $\phi_T$.

$$V_{leading} = \frac{1}{N} \sum_{i=1}^{N} V_{leading,i}$$

$$\phi_T = \frac{1}{N} \sum_{i=1}^{N} \phi_{T,i}$$

**Validation Through Vegetation Based on $\phi_T$:**

- $V_{leading,i} = \phi_{T,i}$
- $\phi_{T,i} = V_{leading,i}$
- $\phi_{T,i} = V_{leading,i}$
- $\phi_{T,i} = V_{leading,i}$
- $\phi_{T,i} = V_{leading,i}$
- $\phi_{T,i} = V_{leading,i}$

**standard deviation for $\phi_T$:**

$$\sigma_{\phi_T} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (\phi_{T,i} - \bar{\phi}_T)^2}$$

**95% certainty region:**

$$\bar{\phi}_T \pm 1.96 \frac{\sigma_{\phi_T}}{\sqrt{N}}$$

**Graphical Presentation:**

- Graph showing $\phi_T$ vs. $V_{leading}$
- Graph showing $\phi_T$ vs. $V_{leading}$
- Graph showing $\phi_T$ vs. $V_{leading}$

**References:**

- [Reference 1](#)
- [Reference 2](#)
- [Reference 3](#)
Each cluster of computed points is generated for the particular plant arrangement and is generated by operating the wind tunnel at some percentage of its maximum rpm. There is scatter in these clusters of both \( \alpha \) and \( R_s \). Use the average value of \( \alpha \) and spline fitting to determine values of \( f_{1} \) for each average value of \( R_s \).

\[
R_s = \frac{1}{\alpha} \sum \frac{1}{R_s}
\]

80% Cylindrical Region: \( C_f \) is \( 1.02 \) up to \( 1.25 \) in diameter

CREATE 8TH LOAD POINT SO HAVE LOADS

\[
C_f = \frac{1}{0.0012} \text{ Reynolds Number}
\]
Appendix E—Equation fitting

This appendix presents a method of determining a single equation for a set of curves. The material has been taken straight from Design of Thermal Systems (3rd Edition, 1971, pp. 63-33) by W.F. Stoecker. This method is used in Appendix D to fit a single equation to a family of friction-factor vs. Reynolds-number curves for a single plant.

E.1 Function of two variables

A performance variable of a component is often a function of two other variables, not just one. For example, the pressure rise developed by the centrifugal pump shown in Figure 1 is a function of both speed \( S \) and the flow rate \( Q \).

If a polynomial expression for the pressure rise, \( \Delta p \), is sought in terms of a second-degree equation in \( S \) and \( Q \), separate equations can be written for each of the three curves in Figure 1. Three points on the 30 rad/s curve would provide the constants in the equation

\[
\Delta p_1 = a_1 + b_1 Q + c_1 Q^2
\]  \( (1) \)

Similar equations for the curves for 24 and 16 rad/s speeds are

\[
\Delta p_2 = a_2 + b_2 Q + c_2 Q^2 \quad \Delta p_3 = a_3 + b_3 Q + c_3 Q^2 \]  \( (2 \& 3) \)

Next the \( a \) constants can be as a second degree equation in terms of \( S \), using the three data points \( a_1 = 30, a_2 = 24, a_3 = 16 \). Such an equation would have the form
\[ a = A_0 + A_1S + A_2S^2 \]  
\[ (4) \]

Similarly for \( b \) and \( c \)

\[ b = B_0 + B_1S + B_2S^2 \quad c = C_0 + C_1S + C_2S^2 \]  
\[ (5 \ & 6) \]

Finally, the constants of Equations 4 through 6 are put into the general equation

\[ \Delta p_1 = (A_0 + A_1S + A_2S^2) + (B_0 + B_1S + B_2S^2)Q + (C_0 + C_1S + C_2S^2)Q^2 \]  
\[ (7) \]

The nine constants \( A, B, C \) can be computed if nine data points from Figure 1 are available.

**E.2 Example**

Manufactures of cooling towers often present catalog data showing the outlet-water temperature as a function of the wet-bulb temperature of the ambient air and the range. The range is the difference between the inlet and outlet temperatures of the water. In Table 1, for example, when the wet-bulb temperature is \( 20^\circ C \) and range is \( 10^\circ C \), the temperature leaving the water is \( 25.9^\circ C \), and so the temperature of the entering water is \( 25.9 + 10 = 35.9^\circ C \). Express the outlet-water temperature \( t \) in Table 1 as a function of the wet-bulb temperature, \( WBT \), and the range, \( R \).

**Solution:** Second-degree polynomial equations in both independent variables will be chosen as the form of the equation. Thus, an equation of the form of Equation 7 can be composed as

\[ t = c_1 + c_2R + c_3R^2 + c_4WBT + c_5(WBT)R \]
\[ + c_6(WBT)^2 + c_7(WBT)^2 + c_8(WBT)^2R + c_9(WBT)^2R^2 \]

| Table 1—Outlet-water temperature, \(^\circ C\), of cooling tower | Wet-bulb temperature (WBT), \(^\circ C\) |
|---|---|---|
| **Range** \((R)\), \(^\circ C\) | **20** | **23** | **26** |
| 10 | \( t = 25.9 \) | 27.5 | 29.4 |
| 16 | 27.0 | 28.4 | 30.2 |
| 22 | 28.4 | 29.6 | 31.3 |
Thus, the equation for \( t \) is

\[
T = 15.2518 + 0.723148R + 0.004167R^2 + 0.325926(WTB)R
+ 0(WTB)^2 + 0.007407(WTB)^2 + 0.000926(WTB)^2R + 0(WTB)^2R^2
\]